Inhomogeneous Cosmological models, what are they good for?

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If the $\Lambda$CDM paradigm model fits observations so well, then why bother looking for alternatives?

- Because “dark matter & “dark energy” are “black boxes” whose detection has been very elusive.
- Because what we have is just

\[
\text{GR} + \text{FLRW} + \text{linear perturbations} + \text{CDM} + \Lambda
\]

Observations are well fit

The converse of this implication is \textbf{NOT} (necessarily) true: fitting observations DOES NOT IMPLY the $\Lambda$CDM model as long as we don’t know the fundamental nature of “DM” & “DE” there is justification in trying to fit observations with other models or even other gravity theories.
We live in the midst of a scientific controversy:

The Orthodoxy says

Your gravity theory is wrong Idiot!!

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HOPEFULLY, there's no need to solve this controversy in the ring!
A “conservative” set of assumptions to fit cosmic observations outside the $\Lambda$-CDM model is to

Keep GR as gravity theory

CDM (or some form of Dark Matter) exists

BUT assume that

$\Lambda = 0$ (or there is no Dark Energy)

FLRW model with linear perturbations DOES NOT provide an appropriate large scale description of Cosmic Dynamics

The Universe should be inhomogeneous at large scales

Observations should be fit by inhomogeneous GR models WITHOUT assuming $\Lambda > 0$ or any form of DE

This set of assumptions MUST BE TESTED
How to construct Inhomogeneous Cosmological models in general

Take a spacetime manifold \((\mathcal{M}, g)\)
- Satisfies Einstein’s equations
  \[ G^{ab} = 8\pi T^{ab} \quad \nabla_b T^{ab} = 0 \]
- Admits a 4-velocity field \(\mathcal{U}^a\)

Define a class of Cosmic Observers

Comoving with 4-velocity \(\mathcal{U}^a\)
- "Time derivative" is \(\frac{d}{d\tau} = \mathcal{U}^a \nabla_a\)

Observers define spatial metric
- "Spatial gradients" are \(\tilde{\nabla}_a = h^{ab}_\alpha \nabla_b\)
General cosmological model is too difficult. We need further assumptions for a CDM dominated universe

- Negligible energy flux and viscous stress
  \[ \Pi_{ab} \approx 0, \quad q_a \approx 0, \]
  \[ \rho \approx mnc^2, \quad p \approx mn\langle v^2 \rangle \ll \rho \]

- Negligible pressure

- Negligible pressure gradients
  \[ h^b_a \nabla_b (p) \ll h^b_a \nabla_b (\rho) \quad \Rightarrow \quad \dot{u}_a \approx 0 \]
  \[ \omega_{ab} \approx 0 \]

- Negligible vorticity (rotation)

- Negligible vector & tensor modes (magnetic fields & gravitational waves)
  \[ H_{ab} \approx 0 \quad \text{Weyl tensor is electric} \]

Dynamics reduces to scalar modes

Inhomogeneous dust universes: dynamical system on

- CDM density & Hubble scalar (common to FLRW)
  \[ \rho, \quad \mathcal{H} = \frac{\Theta}{3} \]

- Electric Weyl & shear tensors (absent in FLRW)
  \[ E_{ab}, \quad \sigma_{ab}, \]

- Deviation from homogeneity

FLRW models follow if

Dynamical System on

- \[ E_{ab} = \sigma_{ab} = 0 \]
- \[ \rho, \quad \mathcal{H} \]
Hierarchy of known exact solutions

**Szekeres models: non-spherical (dipole-like)**

\[ ds^2 = -dt^2 + a^2 \left[ \frac{\Gamma^2 dr^2}{1-k_0 r^2} + \frac{r^2 (dx^2 + dy^2)}{F^2} \right] \]

As LTB  \( a = a(t, r) \)  but  \( \Gamma = \Gamma(t, r, x, y), \ F = F(r, x, y) \)

all quantities depend on \((t, r, x, y)\) in the form  \( A = A_1(t, r) + A_2(t, r, x, y) \)

**LTB models: spherical inhomogeneity**

\[ ds^2 = -dt^2 + a^2 \left[ \frac{\Gamma^2 dr^2}{1-k_0 r^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) \right] \]

two scale factors  \( a = a(t, r), \ \Gamma = \Gamma(t, r) \)

\( \rho(t, r) = \frac{\rho_0(r)}{a^3 \Gamma}, \ \mathcal{H}(t, r) = \frac{\dot{a}}{a} + \frac{\dot{\Gamma}}{3\Gamma} \)

**FLRW models: homogeneous**

\[ ds^2 = -dt^2 + a^2 \left[ \frac{dr^2}{1-k_0 r^2} + r^2 (d\vartheta^2 + \sin^2 \vartheta d\phi^2) \right], \]

one scale factor  \( a = a(t) \)

\( \rho = \rho(t) = \rho_0 a^{-3}, \ \mathcal{H} = \mathcal{H}(t) = \dot{a}/a \)
Why inhomogeneous models with $\Lambda = 0$ may fit cosmic observations?

Because large scale observations is information transmitted by NULL GEODESICS through our past light cone, and ALL the latter is very different for inhomogeneous models.

FLRW model

Relation $d = d(z)$ only fits data if $\Lambda > 0$ (accelerated expansion)

Free parameters $H_0, \Omega_c^m, \Omega_r^\Lambda$

Null Geodesic

LTB model with $\Lambda = 0$

Relation $d = d(z)$ may fit data with $\Lambda = 0$ for certain density profiles (voids)

Free parameters $H_0(r), \Omega_c^m(r), \Omega_r^K(r)$
The Hubble diagram & z-distance module relation for single Gpc size void:

\[ m - M = 5 \log D_L(z) \]

\[ D_L(z) \approx \left( \frac{dD_L}{dz} \right)_0 z + \frac{1}{2} \left( \frac{d^2D_L}{dz^2} \right)_0 z^2 \]

**Compare coefficients: BOTH FIT**

**ΛCDM**

\[ \left( \frac{dD_L}{dz} \right)_0 = \frac{c}{H_0} \]

\[ \frac{1}{2} \left( \frac{d^2D_L}{dz^2} \right)_0 = \frac{c}{4H_0} \left( 2 - \Omega_m + 2\Omega_\Lambda \right) \]

**LTB with Λ = 0**

\[ \left( \frac{dD_L}{dz} \right)_0 = \frac{c}{H_0} \]

\[ \frac{1}{2} \left( \frac{d^2D_L}{dz^2} \right)_0 = \frac{c}{4H_0} \left[ 1 + f(\Omega_0^m(r), \Omega_0^k(r)) \right] \]
“Swiss Cheese” model: simple pattern of “distributed” inhomogeneities in the Universe

The cheese holes are the void regions

We could be inside of one of these void regions
How to make Swiss cheese models?

Represent this by this simple model

Copernicus principle with a larger homogeneity scale

Spherically symmetric dust underdensity (LTB)

The “cheese” is homogeneous dust (FLRW)
Only Gpc size voids seem to fit SN at higher z

300 Mpc Voids in open background

EDS

LCDM

Single Gpc void

$z_{\text{jump}}=0.085$; $\delta_{\text{CENTRE}}=-0.48$
Spherical inhomogeneity is problematic & restrictive: CMB is almost isotropic, so fitting it with an LTB model requires being “near” the center of the void (fine tuning).

1 Gpc

Constraints on amplitude of CMB anisotropies
=> we must be here

Departure from spherical symmetry suggests that this “center problem” can be removed (or made less binding).
The Kinematic Sunyaev Zeldovich effect

Us Today

Relation between CMB Temperature distortion and peculiar velocities of galaxy clusters

Observers here should detect a Large Dipole

Conditions here are very different

CMB (LSS)
Gpc size Spherical Voids:
Single void occupies all observable Universe [check]

Must comply with several observational tests:

- SN Ia
- CMB amplitude & multipoles & BAO
- Initial conditions (LSS) compatible with inflation
- age constraints & H0 measurement
- kinematic S-Z
- etc,

Not Possible !

LTB models are too simple
They lack dynamical freedom
How stringent is the KSZ effect?

Looking the void in the eyes - the kSZ effect in LTB models, Juan García-Bellido & Troels Haugbolle, JCAP, [arXiv:0807.1326]

See also


It has been tested ONLY on spherical LTB models: it does NOT rule out general inhomogeneity (more work needed)
So, let us go beyond spherical voids

A Swiss cheese model, but the “inside” of the holes is no longer spherically symmetric

The “cheese” is homogeneous dust (FLRW)

Dust underdensities (voids) that are NOT spherically symmetric (Szekeres)
Our cosmography at scales < 300 Mpc is obviously NOT spherically symmetric !!!
We try to model this Cosmography with the Szekeres solution


Cross section (tessellation) of the Szekeres density at the “equator”
Coarse-graining cosmic structure by Szekeres solutions

K. Bolejko
Structure formation in the quasispherical Szekeres model
Models of voids & overdensities that “interact”

Figure 4. Density distribution in the considered structure. Upper left panel presents colour coded density distribution of the equatorial cross section (see Fig. 3, bottom panels). Lower left panel presents the vertical cross-section of $X = 0$, through the considered model. The yellow lines correspond to the density profiles, which are presented on the right side. For detailed description see Sec. 6.

Figure 5. The density profile for different time instants: $a = 1$ Gy after the Big Bang, $b = 5.5$ Gy, $c = 10$ Gy, $d$ — present instant.
Without spherical symmetry: the “center” position is no longer unique --- NO need to do “fine-tuning”

In Szekeres quasi-spherical geometry ---- 2 possible “center” locations whose position & orientation changes with time:

\[
A = 4\pi \Phi^2(t, r)
\]

\[
\Phi(t, r_b) = 250 \text{ Mpc}
\]

K. Bolejko & R. A. Sussman

Local isotropic observer where shear vanishes \((r = 0)\)

Positions & orientations change with time

Geometric center of 2-sphere of radius Phi = 250 Mpc
What needs to be done??

Integrate null geodesics for the Szekeres Swiss cheese, and verify the fitting of SN Ia & CMB data

Test the KSZ effect with Szekeres

Difficult because there are no “radial null geodesics”

Much harder work !! (likely will not be done)
Current status in the use of inhomogeneous models to explain cosmic acceleration.

- Spherical Gpc voids are practically ruled out.
- Szekeres voids improve the fitting of observations but perhaps not much (must be tested).
- More general inhomogeneity requires 3d numerical codes & (likely) include small corrections from non-adiabatic and vector/tensor modes.
- There is a consensus in the community that inhomogeneity is no longer favored.
There are other ideas “floating” in the literature

Include radiation. **Effect:** modifies initial conditions, may have effects on CMB fitting

Woei Chet Lim, Marco Regis, Chris Clarkson, JCAP 10 (2013) 010 [arXiv:1308.0902]

Perturbations on an LTB background. **Effect:** allows for a more consistent probing of inhomogeneity

February et al, Class. Quantum Grav. 31 (2014) 175008
inhomogeneity in velocities instead of in densities?

Consider effects of peculiar with respect to the Hubble flow. Effects: KSZ becomes more nuanced


C Tsagas, Peculiar motions, accelerated expansion and the cosmological axis, Phys. Rev. D 84, 063503 (2011)

other ideas?

Consider averages & coarse graining.

If inhomogeneous models cannot explain cosmic acceleration, then why do we need them?

We have become too fixed on the idea that considering inhomogeneous models IMPLIES refuting Dark Energy or $\Lambda$

However: the Universe can still be inhomogeneous with $\Lambda > 0$

NOTICE: the fact that $\Lambda > 0$ does NOT imply a Lambda-CDM Universe
Inhomogeneous models (with $\Lambda > 0$) can still be useful for tackling many problems.

- Check if observations can be fit with an inhomogeneous model with $\Lambda > 0$ (a $\Lambda$-LTB model). Marra et al:

- Observational effect of inhomogeneities: they give the false “impression” of a varying $\Lambda$. Romano et al:
  - Non perturbative effects of primordial curvature perturbations on the apparent value of a cosmological constant, Antonio Enea Romano, Sergio Sanes, Misao Sasaki, Alexei A. Starobinsky EPL, [arXiv:1311.1476]

- Theoretical issues: non-locality and averaging,

- Better understanding of alternative gravity theory
Provide a theoretical framework for non-linear perturbations

Examine relativistic & non-linear effects in structure formation and growth suppression

Exact solutions as “exact” perturbations: look at the following hierarchy
Propose a solution based on assuming “EXACT” perturbation forms:

\[
\rho = \rho_q \left[ 1 + \delta^{(\rho)} \right], \quad \mathcal{H} = \mathcal{H}_q \left[ 1 + \delta^{(\mathcal{H})} \right]
\]

where: \( \{ \rho_q, \mathcal{H}_q \} \) are SZEKERES scalars that satisfy FLRW dynamics

and: \( \{ \delta^{(\rho)}, \delta^{(\mathcal{H})} \} \) are obtained from the 1+3 system

Look at the dynamics of these perturbations & compare with “standard” perturbations
The perturbations compare local covariant scalars with their weighed average that satisfies FLRW dynamics.

\[ A_q = \frac{\int_D A \mathcal{F} d\mathcal{V}}{\int_D \mathcal{F} d\mathcal{V}} \]

\[ d\mathcal{V} = \sqrt{\det(h_{ab})} \, d^3x \]

Proper volume

Average with weight factor

\[ \mathcal{F} = \left[ \dot{R}^2 + \left(1 - \frac{2M}{R}\right)\right]^{1/2} \]

FLRW background defined in terms of averaged scalars

Foliation by spherical comoving domains

\[ D[r] = \psi[r] \times S^2(\theta, \phi) \]

Comparison between local value \( A \) and weighed average \( A_q \) at each 2-sphere

\[ \delta(A) = \frac{A - A_q}{A_q} \]

Proper volume
We transform Szekeres dynamics into evolution equations for EXACT & COVARIANT perturbations on FLRW:

\begin{align*}
\dot{\rho}_q &= -3 \rho_q H_q, \\
\dot{H}_q &= -H^2 - \frac{4\pi}{3} \rho_q,
\end{align*}

\begin{align*}
\dot{\delta}(\rho) &= -3 \left(1 + \delta(\rho)\right) H_q \delta(H) \\
\dot{\delta}(H) &= - \left[ \left(1 + 3\delta(H)\right) \delta(H) - \frac{\Omega_q}{2} \left(\delta(H) - \delta(\rho)\right) \right] H_q,
\end{align*}

\begin{align*}
H_q^2 &= \frac{8\pi}{3} \rho_q - K_q, \\
\Omega_q &= \frac{8\pi \rho_q}{3 H_q^2}, \\
2 \delta(H) &= \Omega_q \delta(\rho) + (1 - \Omega_q) \delta(K) \\
\delta(\Omega) &= \delta(\rho) - 2\delta(H).
\end{align*}

Algebraic constraints: \quad \Rightarrow \quad \text{Autonomous ODE's: DYNAMICAL SYSTEM !!}

Reduce to standard perturbations in the linear limit.
Growth suppression factor

Linear perturbations on FLRW

The growth suppression factor defined for linear perturbations (in an isochronous gauge) on a dust FLRW background with $\Lambda > 0$ or $\Lambda = 0$ is

\[
f = \frac{d(\ln \delta)}{d(\ln \bar{a})} = \frac{(\ln \bar{a})'}{\ln \bar{a}} = \frac{\dot{\delta}/\delta}{\dot{\bar{a}}/\bar{a}},
\]

(1)

\[
\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}, \quad \dot{\bar{a}} = \frac{\Theta}{3} = \bar{H},
\]

(2)

where $\bar{a}$, $\bar{\rho}$, $\bar{H}$ are the scale factor, density and Hubble scalar of the FLRW background and $\delta$ is the density contrast satisfying the linear equation

\[
\ddot{\delta} + 2\bar{H}\dot{\delta} - 4\pi\bar{\rho}\delta = 0.
\]

(3)

Exact dust perturbations (LTB & Szekeres)

\[
\ddot{\delta}_q^{(\rho)} - \frac{2}{1 + \delta_q^{(\rho)}} \left[ \delta_q^{(\rho)} \right]^2 + 2H_q\dot{\delta}_q^{(\rho)} - 4\pi\rho_q\delta_q^{(\rho)}(1 + \delta_q^{(\rho)}) = 0,
\]

Relation between exact perturbations vs curvature & kinematic invariants

\[
\sigma_{ab} = \Sigma e_{ab}, \quad \Sigma = -(H - H_q) = -H_q\delta_q^{(H)},
\]

\[
E_{ab} = \Psi_2 e_{ab}, \quad \Psi_2 = \frac{4\pi}{3} (\rho - \rho_q) = \frac{4\pi}{3} \rho_q\delta_q^{(\rho)},
\]

$\mathcal{R}$ Ricci Scalar $\Psi_2$ Weyl conformal invariant

Invariant meaning of growth suppression factor

\[
f = -\frac{3\xi}{\phi} = -\frac{\Sigma/H}{2\Psi_2/\mathcal{R}},
\]

Ratio: anisotropy of expansion vs Weyl/Ricci curvature
Numerical results for 50 Mpc LTB voids

$\Omega^m = 0.25, \Omega^\Lambda = 0$

$\Omega^m = 0.25, \Omega^\Lambda = 0.75$

$\Lambda$ introduces a strong suppression effect, but may not be noticeable in our cosmic time $t = t_0$ (more discussion needed)
THANKS FOR YOUR ATTENTION