A generalized wave equation and its application to dark matter halos

Pierre-Henri Chavanis

Université Paul Sabatier & CNRS

Laboratoire de Physique Théorique - Toulouse, France

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Plan of the presentation

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Complex hydrodynamic representation of the Schrödinger equation

We consider the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi \tag{1}$$

for a quantum particle in a potential Φ. If we make the WKB transformation

$$\psi = e^{i\mathcal{S}/\hbar}, \tag{2}$$

where $\mathcal{S}$ is a complex action, we obtain the quantum Hamilton-Jacobi equation

$$\frac{\partial\mathcal{S}}{\partial t} + \frac{1}{2m}(\nabla\mathcal{S})^2 + m\Phi = i\frac{\hbar}{2m}\Delta\mathcal{S}. \tag{3}$$

When $\hbar = 0$ we recover the classical Hamilton-Jacobi equation (in that case $\mathcal{S}$ is real).

Complex hydrodynamic representation of the Schrödinger equation

If we introduce the complex velocity field

$$\mathbf{U} = \frac{\nabla \mathcal{S}}{m}, \tag{4}$$

and take the gradient of the quantum Hamilton-Jacobi equation, we obtain

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = i\frac{\hbar}{2m}\Delta\mathbf{U} - \nabla\Phi. \tag{5}$$

This equation can be interpreted as a viscous Burgers equation (no pressure term) in fluid mechanics involving a complex velocity field and an imaginary viscosity,

$$\nu = \frac{i\hbar}{2m}, \tag{6}$$

proportional to the Planck constant and inversely proportional to the mass of the particle. Therefore, quantum mechanics may be interpreted as a generalized hydrodynamics involving a complex velocity field and an imaginary viscosity.

Complex hydrodynamic representation of the Schrödinger equation

The complex hydrodynamic equation (5) can be written in the form of Newton's law

$$\frac{D\mathbf{U}}{Dt} = -\nabla\Phi, \tag{7}$$

where

$$\frac{D\mathbf{U}}{Dt} = \frac{\partial\mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} - i\mathcal{D}\Delta\mathbf{U} \tag{8}$$

is a complex acceleration and

$$\mathcal{D} = \frac{\hbar}{2m} \tag{9}$$

is a quantum diffusion coefficient. Eq. (7) can be interpreted as a complex equation of dynamics. The Schrödinger equation can be obtained from Newton's law by replacing the velocity by a complex velocity field and the time derivative by a generalized advection operator.

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Derivation of a generalized Schrödinger equation

In order to take dissipative effects into account, we would naively write the fundamental equation of dynamics under the form

$$\frac{D\mathbf{U}}{Dt} = -\nabla\Phi - \gamma\mathbf{U}, \tag{10}$$

where γ is a complex friction coefficient. However, it can be shown that this equation leads to a generalized Schrödinger equation that does not conserve the normalization condition locally.

Derivation of a generalized Schrödinger equation

In order to take dissipative effects into account, while conserving the normalization condition locally, we write the fundamental equation of dynamics under the form

$$\frac{D\mathbf{U}}{Dt} = -\nabla\Phi - \mathrm{Re}(\gamma\mathbf{U}), \tag{11}$$

where γ is a complex friction coefficient. Using the expression (8) of the complex acceleration, Eq. (11) can be rewritten as a damped complex viscous Burgers equation of the form

$$\frac{\partial\mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = i\frac{\hbar}{2m}\Delta\mathbf{U} - \nabla\Phi - \mathrm{Re}(\gamma\mathbf{U}). \tag{12}$$

Derivation of a generalized Schrödinger equation

We assume that the complex velocity field is potential so it can be written as the gradient of a function :

$$\mathbf{U} = \frac{\nabla \mathcal{S}}{m}. \tag{13}$$

As a consequence of Eq. (13), the flow is irrotational : $\nabla \times \mathbf{U} = \mathbf{0}$. Using the well-known identities of fluid mechanics $(\mathbf{U} \cdot \nabla)\mathbf{U} = \nabla(\mathbf{U}^2/2) - \mathbf{U} \times (\nabla \times \mathbf{U})$ and $\Delta \mathbf{U} = \nabla(\nabla \cdot \mathbf{U}) - \nabla \times (\nabla \times \mathbf{U})$ which reduce to $(\mathbf{U} \cdot \nabla)\mathbf{U} = \nabla(\mathbf{U}^2/2)$ and $\Delta \mathbf{U} = \nabla(\nabla \cdot \mathbf{U})$ for an irrotational flow, and using the identity $\nabla \cdot \mathbf{U} = \Delta \mathcal{S}/m$ resulting from Eq. (13), we find that Eq. (12) can be integrated into

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2m}(\nabla \mathcal{S})^2 - i\frac{\hbar}{2m}\Delta \mathcal{S} + m\Phi + \mathrm{Re}(\gamma \mathcal{S}) = 0. \tag{14}$$

Equation (14) can be viewed as a damped quantum Hamilton-Jacobi equation for a complex action, or as a damped Bernoulli equation for a complex potential.

Derivation of a generalized Schrödinger equation

We define the wave function $\psi(\mathbf{r}, t)$ through the complex Cole-Hopf transformation

$$\mathcal{S} = -i\hbar \ln \psi, \tag{15}$$

Equation (15) can be rewritten in the WKB form

$$\psi = e^{i\mathcal{S}/\hbar} \tag{16}$$

relating the wavefunction to the complex action. Therefore, the complex Cole-Hopf transformation (15) is equivalent to the WKB transformation (16). Substituting Eq. (15) into Eq. (14), and using the identity

$$\Delta(\ln \psi) = \frac{\Delta\psi}{\psi} - \frac{1}{\psi^2}(\nabla\psi)^2, \tag{17}$$

we obtain the generalized Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + \hbar\,\mathrm{Im}(\gamma \ln \psi)\psi. \tag{18}$$

Derivation of a generalized Schrödinger equation

Writing $\gamma = \gamma_R + i\gamma_I$, where γ_R is the classical friction coefficient and γ_I is the quantum friction coefficient, and using the identity

$$\mathrm{Im}(\gamma \ln \psi) = \gamma_I \ln|\psi| - \frac{1}{2} i\gamma_R \ln\left(\frac{\psi}{\psi^*}\right), \qquad (19)$$

we can rewrite Eq. (18) in the equivalent form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + \hbar\gamma_I \ln|\psi|\,\psi - i\frac{\hbar}{2}\gamma_R \ln\left(\frac{\psi}{\psi^*}\right)\psi. \qquad (20)$$

Introducing the notations

$$\gamma_R = \xi, \qquad \gamma_I = \frac{2k_B T}{\hbar}, \qquad (21)$$

the generalized Schrödinger equation (20) takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + 2k_B T \ln|\psi|\,\psi - i\frac{\hbar}{2}\xi \ln\left(\frac{\psi}{\psi^*}\right)\psi. \qquad (22)$$

Using the hydrodynamic (Madelung) representation of the generalized Schrödinger equation (22), we will see that ξ plays the role of an ordinary friction coefficient while T plays the role of an effective temperature.

Derivation of a generalized Schrödinger equation

It is interesting to note that the complex nature of the friction coefficient

$$\gamma = \xi + i\frac{2k_B T}{\hbar} \tag{23}$$

leads to a generalized Schrödinger equation exhibiting *simultaneously* a friction term (as expected) and an effective temperature term (unexpected). They correspond to the real and imaginary parts of γ. This may be viewed as a new form of fluctuation-dissipation theorem. In this respect, we note that the relation

$$\mathcal{D} = \frac{k_B T}{m\gamma_I} \tag{24}$$

resulting from Eqs. (9) and (21) is similar to the Einstein relation of Brownian motion.

Derivation of a generalized Schrödinger equation

On the other hand, if we assume that $\gamma_{\rm R} = \gamma_{\rm I}$ we obtain the relation

$$\mathcal{D} = \frac{k_B T}{m\xi}. \tag{25}$$

Explicitly,

$$\frac{\hbar}{2m} = \frac{k_B T}{m\xi} \qquad \text{or} \qquad \frac{\hbar}{2} = \frac{k_B T}{\xi}. \tag{26}$$

Again, this can be viewed as a sort of generalized Einstein relation expressing a form of fluctuation-dissipation theorem. Here, the diffusion coefficient has a quantum origin.

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The Madelung transformation

We consider a self-gravitating BEC described by the Gross-Pitaevskii-Poisson (GPP) equations

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+m\Phi\psi+\frac{4\pi a_s\hbar^2}{m^2}|\psi|^2\psi, \tag{27}$$

$$\Delta\Phi=4\pi G|\psi|^2, \tag{28}$$

where a_s is the scattering length of the bosons. This has been proposed as a model of dark matter halos. The density is $\rho=|\psi|^2$, where ψ is the wave function of the condensate.

The Madelung transformation

Let us consider the generalized GPP equations

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+m\Phi\psi+\frac{4\pi a_s\hbar^2}{m^2}|\psi|^2\psi+2k_B T\ln|\psi|\psi-i\frac{\hbar}{2}\xi\ln\left(\frac{\psi}{\psi^*}\right)\psi, \quad (29)$$

$$\Delta\Phi=4\pi G|\psi|^2. \quad (30)$$

In order to enlighten the physical meaning of the generalized GPP equations (29) and (30), we can write them in the form of hydrodynamic equations by using the Madelung transformation.

The Madelung transformation

We write the wavefunction as

$$\psi(\mathbf{r},t)=\sqrt{\rho(\mathbf{r},t)}\,e^{iS(\mathbf{r},t)/\hbar}, \tag{31}$$

where $\rho(\mathbf{r},t)=|\psi|^2$ is the density and $S(\mathbf{r},t)$ is the real action. Following Madelung, we introduce the velocity field

$$\mathbf{u}=\frac{\nabla S}{m}. \tag{32}$$

Since the velocity is potential, the flow is irrotational : $\nabla\times\mathbf{u}=\mathbf{0}$.

The Madelung transformation

Substituting Eq. (31) into the generalized GPP equation and separating real and imaginary parts, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{33}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + m\left[\Phi + h(\rho)\right] + Q + \xi S = 0, \tag{34}$$

where

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{35}$$

is the quantum potential which takes into account the Heisenberg uncertainty principle and $h(\rho) = (4\pi a_s \hbar^2/m^3)\rho + (k_B T/m)\ln\rho$ is a potential which takes into account the nonlinearities in the generalized GPP equation. The first equation is similar to the equation of continuity in hydrodynamics. The second equation has a form similar to the classical Hamilton-Jacobi equation with an additional quantum potential and a source of dissipation. It can also be interpreted as a generalized Bernoulli equation for a potential flow.

The Madelung transformation

Taking the gradient of Eq. (34), and using the well-known identity of vector analysis $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\mathbf{u}^2/2) - \mathbf{u}\times(\nabla\times\mathbf{u})$ which reduces to $(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\mathbf{u}^2/2)$ for an irrotational flow, we obtain an equation similar to the Euler equation with a linear friction and a quantum force

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi - \frac{1}{m}\nabla Q - \xi\mathbf{u}, \tag{36}$$

where

$$P = \frac{2\pi a_s\hbar^2}{m^3}\rho^2 + \rho\frac{k_B T}{m} \tag{37}$$

is the total pressure. It is the sum of a quadratic equation of state due to the self-interaction of the bosons and an isothermal (linear) equation of state due to the logarithmic term in the generalized GP equation (29).

The Madelung transformation

In conclusion, the generalized GPP equations are equivalent to the hydrodynamic equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{38}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi - \frac{1}{m}\nabla Q - \xi \mathbf{u}, \tag{39}$$

$$\Delta \Phi = 4\pi G \rho. \tag{40}$$

Equation (41) is the continuity equation, Eq. (42) is the momentum equation, and Eq. (43) is the Poisson equation. We note that the momentum equation involves a damping term, proportional and opposite to the velocity, corresponding to the last term in the generalized GP equation (29). We shall refer to these equations as the damped quantum barotropic Euler-Poisson equations.

The Madelung transformation

For dissipationless systems ($\xi = 0$), they reduce to the quantum barotropic Euler-Poisson equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{41}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla \Phi - \frac{1}{m}\nabla Q, \tag{42}$$

$$\Delta \Phi = 4\pi G \rho. \tag{43}$$

When the quantum potential can be neglected (TF approximation), we recover the classical barotropic Euler-Poisson equations.

Connection with the equations of Brownian theory

In the overdamped limit $\xi \to +\infty$, we can formally neglect the inertial term in the Euler equation (42) so that

$$\xi \mathbf{u} \simeq -\frac{1}{\rho}\nabla P - \nabla\Phi - \frac{1}{m}\nabla Q. \tag{44}$$

Substituting this relation into the continuity equation (41), we obtain the quantum barotropic Smoluchowski-Poisson (SP) equations

$$\xi\frac{\partial\rho}{\partial t} = \nabla\cdot\left(\nabla P + \rho\nabla\Phi + \rho\nabla\Phi_{\rm ext} + \frac{\rho}{m}\nabla Q\right), \tag{45}$$

$$\Delta\Phi = 4\pi G\rho. \tag{46}$$

When the quantum potential can be neglected (TF approximation), we recover the classical barotropic SP equation which arises in the context of nonlinear Fokker-Planck equations and generalized thermodynamics.

Connection with the equations of Brownian theory

The isothermal equation of state $P = \rho k_B T/m$ yields an ordinary diffusion term

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi + \rho \nabla \Phi_{\rm ext} + \frac{\rho}{m} \nabla Q \right), \quad (47)$$

with a diffusion coefficient given by the Einstein formula $D = k_B T/\xi m$. The polytropic equation of state $P = K\rho^\gamma$ leads to anomalous diffusion

$$\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left(K \nabla \rho^\gamma + \rho \nabla \Phi + \rho \nabla \Phi_{\rm ext} + \frac{\rho}{m} \nabla Q \right), \quad (48)$$

It is interesting to recover the equations of Brownian theory from the generalized GP equation (29) in the strong friction limit $\xi \to +\infty$. In this sense, the generalized GP equation (29) makes the connection between quantum mechanics and Brownian theory. However, the analogy with Brownian theory is essentially effective.

Plan of the presentation

Fundamental equation of quantum hydrostatic equilibrium

The condition of quantum hydrostatic equilibrium

$$\frac{\rho}{m}\nabla Q + \nabla P + \rho\nabla\Phi = \mathbf{0} \tag{49}$$

describes the balance between the quantum potential arising from the Heisenberg uncertainty principle, the pressure due to short-range interactions (scattering), the pressure due to the effective temperature, and the gravitational attraction. Combining Eq. (49) with the Poisson equation (43), we obtain the fundamental differential equation of quantum hydrostatic equilibrium

$$\frac{\hbar^2}{2m^2}\Delta\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \nabla\cdot\left(\frac{\nabla P}{\rho}\right) = 4\pi G\rho. \tag{50}$$

Fundamental equation of quantum hydrostatic equilibrium

For the equation of state (33), it takes the form

$$\frac{\hbar^2}{2m^2}\Delta\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)-\frac{4\pi a_s\hbar^2}{m^3}\Delta\rho-\frac{k_B T}{m}\Delta\ln\rho=4\pi G\rho. \tag{51}$$

This differential equation determines the general equilibrium density profile $\rho(\mathbf{r})$ of a BECDM halo in our model. This profile generically has a core-halo structure with a solitonic core and an isothermal halo.

Generalized Emden equation

We assume that the bosons have a repulsive self-interaction ($a_s > 0$) and we make the TF approximation ($Q = 0$). In that case, Eq. (51) reduces to

$$-\frac{4\pi a_s \hbar^2}{m^3}\Delta\rho - \frac{k_B T}{m}\Delta \ln \rho = 4\pi G\rho. \tag{52}$$

We write

$$\rho = \rho_0 e^{-\psi} \quad \text{and} \quad \xi = \frac{r}{r_0}, \tag{53}$$

where ρ_0 is the central density and r_0 is the thermal core radius defined by

$$r_0 = \left(\frac{k_B T}{4\pi G \rho_0 m}\right)^{1/2} \tag{54}$$

We also introduce the dimensionless parameter

$$\chi = \frac{4\pi a_s \hbar^2 \rho_0}{m^2 k_B T}, \tag{55}$$

which is a measure of the central density ρ_0 for a given value of the temperature T. We call it the concentration parameter.

Generalized Emden equation

Equation (52) then takes the form of a generalized Emden equation

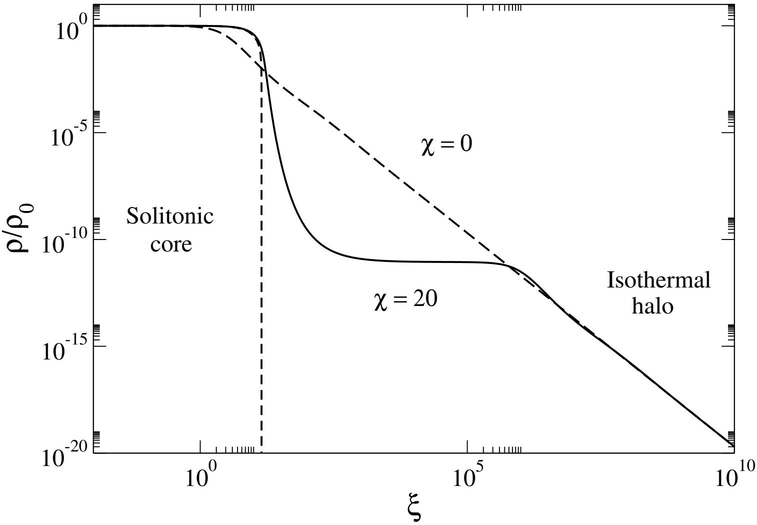
$$\Delta\psi + \chi\nabla\cdot\left(e^{-\psi}\nabla\psi\right) = e^{-\psi}. \tag{56}$$

The ordinary Emden equation is recovered for $\chi = 0$. For a spherically symmetric configuration, the generalized Emden equation (56) takes the form

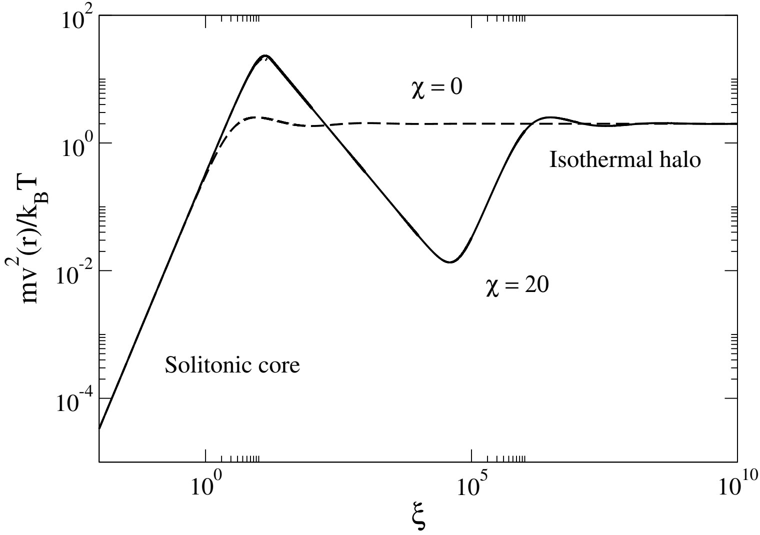
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) + \frac{\chi}{\xi^2}\frac{d}{d\xi}\left(\xi^2 e^{-\psi}\frac{d\psi}{d\xi}\right) = e^{-\psi}. \tag{57}$$

For a given value of χ, this equation can be solved numerically with the boundary conditions $\psi(0) = \psi'(0) = 0$.

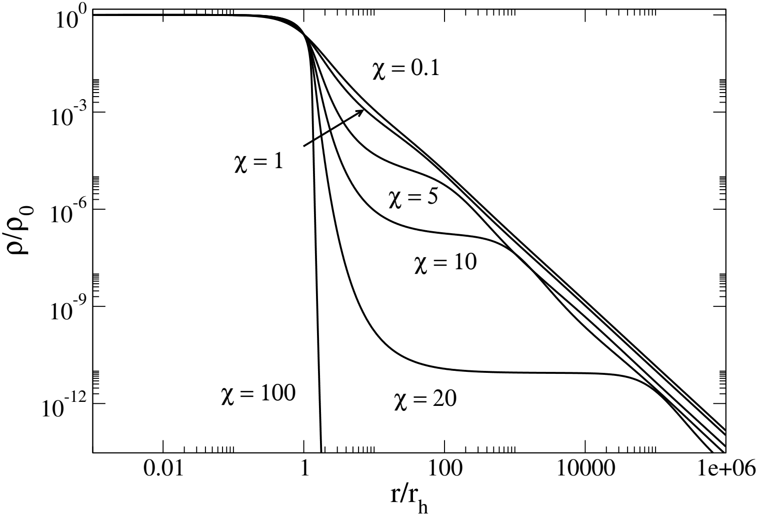
Density profile



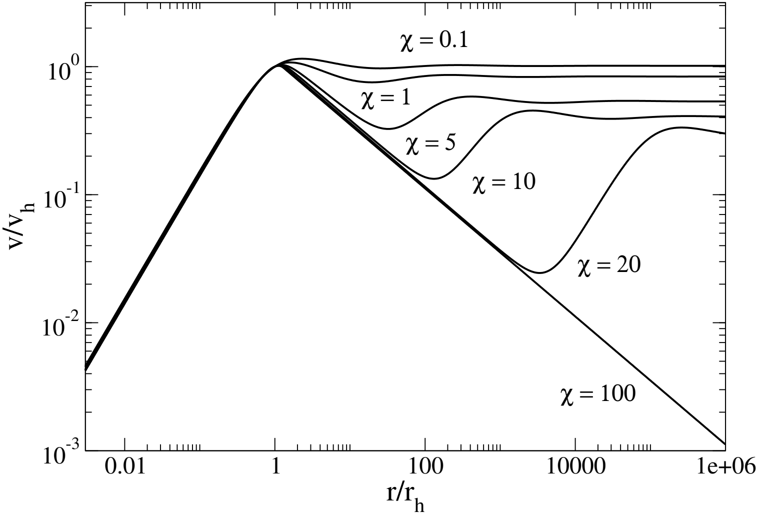
Circular velocity profile



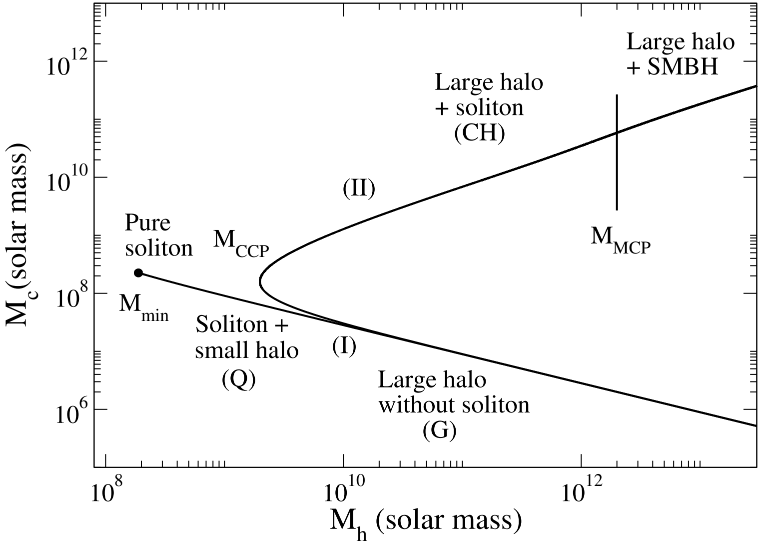
Density profiles



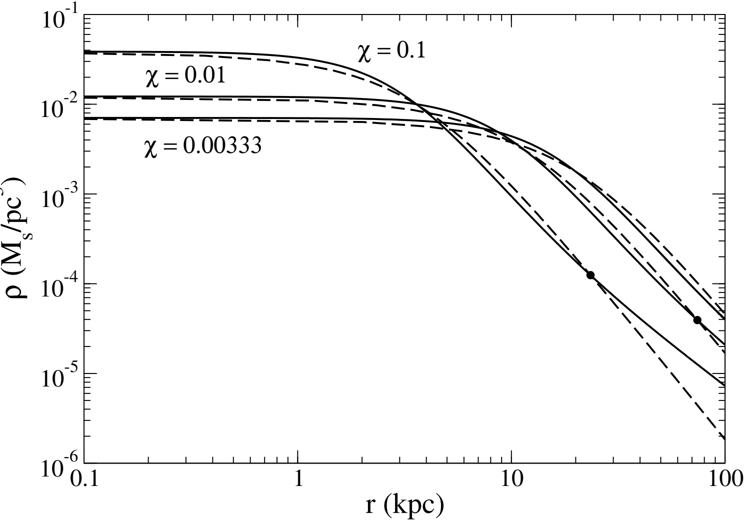
Circular velocity profiles



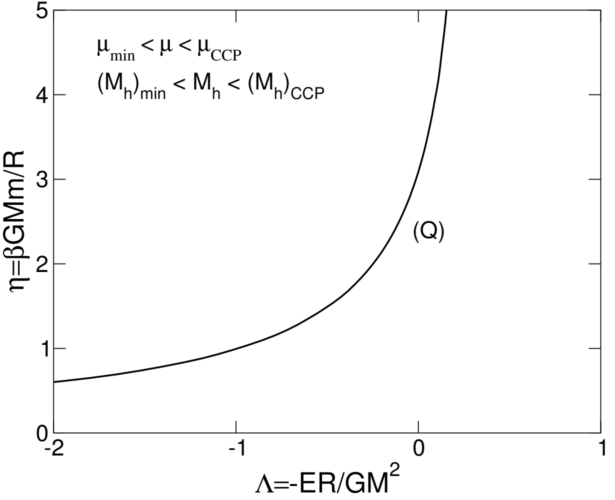
Phase diagram



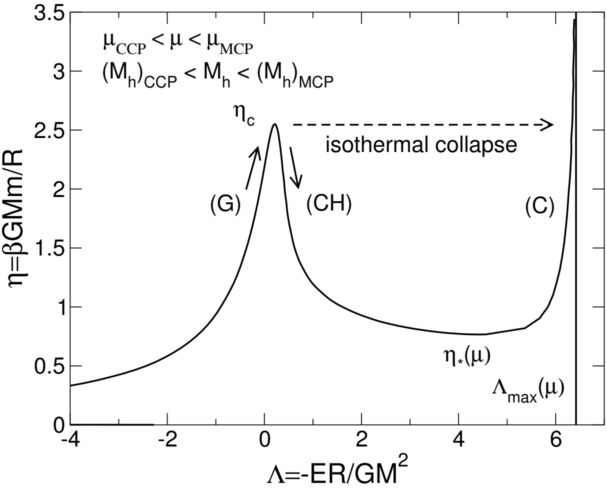
Isothermal and Burkert profiles



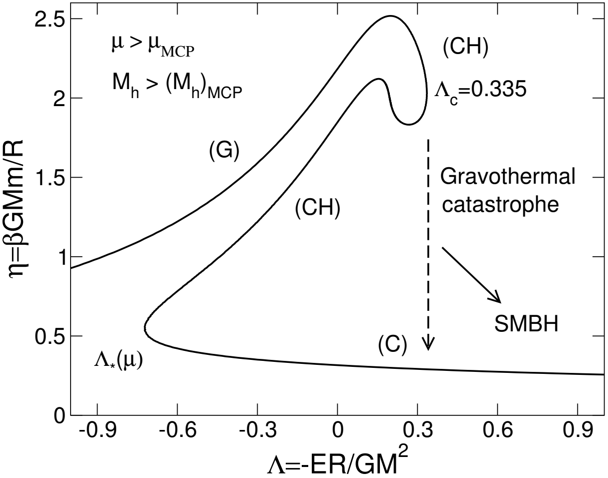
Ultracompact halos : quantum regime



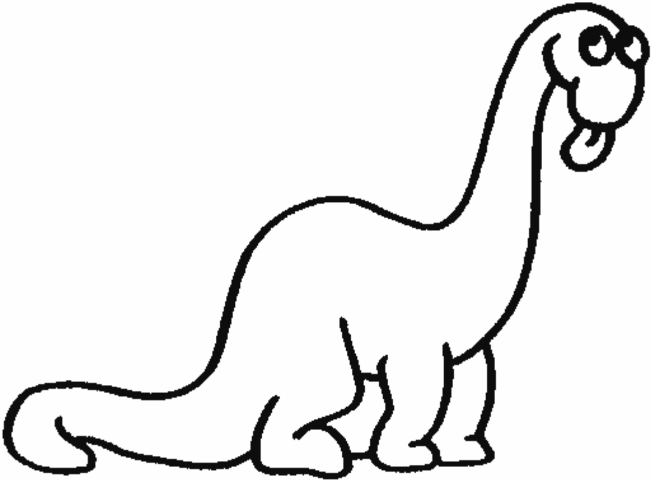
Small halos : core-halo structure



Large halos : black hole formation



Dinosaur's neck



References

See a detailed list of references in :

P.-H. Chavanis, *A predictive model of BEC dark matter halos with a solitonic core and an isothermal atmosphere*, arXiv :1810.08948