Quasinormal modes 2: Spinning black holes

Aaron Zimmerman
XII Mexican School on Gravitation and Mathematical Physics

November 6, 2018
Last time

• Scalar wave equation: \( \Box_g \Phi = -4\pi T \)
• Separation of variables:

\[ \Phi_{\omega lm} \sim e^{-i\omega t} \frac{u_{\omega lm}(r)}{r} Y_{lm}(\theta, \phi) \]

• Radial equation:

\[ \frac{d^2 u_{\omega lm}}{dr_*^2} + (\omega^2 - V) u_{\omega lm} = 0 \]

\[ V = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right) \]
Last time

- For orbiting source with a few freq
  \[
  \frac{d^2 u_{\omega lm}}{dr_*^2} + (\omega^2 - V)u_{\omega lm} = S_{\omega lm}(r)
  \]
  - Decompose source
  - GF via series sln or direct integration
  - Integrate GF over source moments
  - Assemble
  - Or, solve 1+1 D wave eqn with source
Last time

- Outgoing BCs only for special freq
- Leaver’s method gives
  \[ \omega_{lmn} \quad u_{lmn}(r) \quad A_{\text{out}} \]

- Similar series expansions get \( \partial_\omega A_{\text{in}} \)

- This gives GF, use source evaluated at QNMs to get amplitudes and phases modes

- Grav waves treated the same: \( s\psi_{\omega lm} \rightarrow h_{\mu\nu} \)
Part 1

ROTATING BLACK HOLES
Rotating black holes: Kerr

- Parametrized by mass and spin parameter $M, a$
- Horizon at $r_+ = M \left(1 + \sqrt{1 - a^2/M^2}\right)$
- Frame dragging: $g_{t\phi} \neq 0$
- Ergoregion
  \[
  r^2 - 2Mr + a^2 \cos^2 \theta < 0
  \]
- Horizon rotates at $\Omega_H \propto a$
Rotating black holes: Kerr

- Parametrized by mass and spin parameter $M, a$

- Horizon at $r_+ = M(1 + \sqrt{1 - a^2/M^2})$

- Frame dragging: $g_{t\phi} \neq 0$

- Ergoregion

$$r^2 - 2Mr + a^2 \cos^2 \theta < 0$$

- Horizon rotates at $\Omega_H \propto a$

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta}\right) dt^2 + \frac{r^2 + a^2 \cos^2 \theta}{\Delta} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2$$

$$-\frac{4Mar \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dt d\phi + \frac{\sin^2 \theta((r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} d\phi^2$$
Rotating black holes: Kerr

- Parametrized by mass and spin parameter $M$, $a$
- Horizon at $r_+ = M \left(1 + \sqrt{1 - a^2/M^2}\right)$
- Frame dragging: $g_{t\phi} \neq 0$
- Ergoregion
- Horizon rotates at $\Omega_H \propto a$

$$\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$$
Weyl curvature scalar

- For Kerr spacetime, curvature quantities most natural

\[ \Psi_4 = C_{\mu\nu\rho\sigma} n^\mu m^{\nu*} n^\rho m^{\sigma*} \]

- Example: flat space perts \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \)

\[
\begin{align*}
\ell^\mu &= \frac{(\partial_t)^\mu + (\partial_r)^\mu}{\sqrt{2}} \\
n^\mu &= \frac{(\partial_t)^\mu - (\partial_r)^\mu}{\sqrt{2}} \\
m^\mu &= \frac{(\partial_\theta)^\mu + i \csc \theta (\partial_\phi)^\mu}{\sqrt{2}r}
\end{align*}
\]
Weyl curvature scalar

• For Kerr spacetime, curvature quantities most natural

\[ \Psi_4 = C_{\mu\nu\rho\sigma} n^\mu m^{\nu*} n^\rho m^{\sigma*} \]

• Example: flat space perturbations \[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

\[ l^\mu = \frac{(\partial_t)^\mu + (\partial_r)^\mu}{\sqrt{2}} \quad n^\mu = \frac{(\partial_t)^\mu - (\partial_r)^\mu}{\sqrt{2}} \]

\[ m^\mu = \frac{(\partial_\theta)^\mu + i \csc \theta (\partial_\phi)^\mu}{\sqrt{2r}} \]

\[ \Psi_4 = -\ddot{h}_+ + i\ddot{h}_\times \]
Perturbations of Kerr

\[ \Phi \]

\[ A_\mu \rightarrow \Phi_0, \Phi_2 \rightarrow L_T[s \psi] = 0 \]

Project onto basis \( l^\mu, n^\mu, m^\mu, m^*\mu \)

\[ \Psi_0, \Psi_4 \]

Master equation

\[ s \psi_{lm\omega} = e^{-i\omega t + im\phi} s R_{lm\omega}(r) s S_{lm\omega}(\theta) \]

\[ s = -2 : s \psi = \Psi_4 \rightarrow h_{\mu\nu} \]
Black hole perturbation theory

- Angular equation

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d S_{lm\omega}}{d\theta} \right) + V_\theta(\omega, A_{lm}) S_{lm\omega} = 0 \]

- Potential is more complicated than Schw

\[ V_\theta = s E_{lm\omega} - \frac{m^2}{\sin^2 \theta} - s^2 \cot^2 \theta - s^2 + a^2 \omega^2 \cos^2 \theta - 2a \omega s \cos \theta \]

- Solutions are deformations of (spin-weighted) spherical harmonics
Black hole perturbation theory

- Radial equation for $u_{lm\omega} = \Delta^{s/2} \sqrt{r^2 + a^2} R_{lm\omega}$

$$
\frac{d^2 u_{lm\omega}}{dr_*^2} + V_r u_{lm\omega} = S_{lm\omega}(r)
$$

$$
V_r = \left( \omega - \frac{am}{r^2 + a^2} \right)^2 - 2is \frac{r - M}{r^2 + a^2} \left( \omega - \frac{am}{r^2 + a^2} \right)
+ \frac{\Delta}{(r^2 + a^2)^2} (4ir\omega s - \lambda) + F(r, s)
$$

$$
s\lambda_{lm\omega} = sE_{lm\omega} - s(s + 1) + a^2\omega^2 - 2am\omega
$$
Black hole perturbation theory

- Radial equation for $u_{lm\omega} = \Delta^{s/2} \sqrt{r^2 + a^2} R_{lm\omega}$

$$\frac{d^2 u_{lm\omega}}{dr_*^2} + V_r u_{lm\omega} = S_{lm\omega}(r)$$

$$V_r = \left( \omega - \frac{am}{r^2 + a^2} \right)^2 - 2is \frac{r - M}{r^2 + a^2} \left( \omega - \frac{am}{r^2 + a^2} \right)$$

$$+ \frac{\Delta}{(r^2 + a^2)^2} (4ir\omega s - \lambda) + F(r, s)$$

$$r_* = r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln \left( \frac{r - r_+}{r_+} \right) - \frac{r_-^2 + a^2}{r_+ - r_-} \ln \left( \frac{r - r_-}{r_+} \right)$$
Scattering and superradiance

• Scattering of scalar waves as before

\[ u_{in} \sim \begin{cases} 
A_{in}e^{-i\omega r_*} + A_{out}e^{i\omega r_*} & r_* \to \infty \\
e^{-i(\omega - m\Omega_H)r_*} & r_* \to -\infty
\end{cases} \]

• Try to derive flux conservation

\[ W[u_{in}, u_{in}^*] = u_{in} \frac{d u_{in}^*}{dr_*} - u_{in}^* \frac{d u_{in}}{dr_*} \Rightarrow \left(1 - \frac{m\Omega_H}{\omega}\right)|T|^2 = 1 - |R|^2 \]

• More flux out than went in if \( \omega < m\Omega_H \)

• Energy taken from BH: Penrose process for waves
Superradiance (Grav scattering)

\[ Z = \frac{dE_{\text{out}}/dt}{dE_{\text{in}}/dt} \]

QNMs of Kerr

- Photon orbits vary with inclination
- Wave picture: freq and decay split with $m$
- As spin increases, freq increases, decay rate decreases
- Modes determined by mass and spin
Geometric optics (WKB)

\[ S = Ma \]

\[ \Omega H \]

Yang et al (2012)
Geometric optics (WKB)

\[ S = M \alpha \]

\[ \Omega_H \]

\[ \mu \sim \cos \iota \]

\[ \iota \sim 42^\circ \]

Yang et al (2012)
Kerr QNMs

\[ Q = \frac{\omega}{2\gamma} \]

\[ (\omega_R, \gamma) \rightarrow (M, a/M) \]
Schwarzschild QNMs

Leaver (1985)
Splitting of spectrum

\[ \begin{align*}
\text{ZDM0} & : \quad \omega_R, \\
\text{ZDM1} & : \quad \omega_R, \\
\text{ZDM2} & : \quad \omega_R, \\
\text{ZDM3} & : \quad \omega_R. \\
\end{align*} \]
Quasinormal mode response

- Have source-free solutions

\[ \psi_{lm\omega} \sim e^{-i\omega t + im\phi} R_{lm\omega}(r) S_{lm\omega}(\theta) \]

\[ R_{lm\omega} = \Delta^{-s/2}(r^2 + a^2)^{-1/2} u_{lm\omega} \]

- Build response func in time domain

\[ G(x^\mu, x'^\mu) = \frac{1}{2\pi} \sum_{l,m} \int d\omega \ e^{-i\omega(t-t')} g_{lm\omega}(r, r') \Omega(\theta, \theta', \phi, \phi') \]

\[ g(r, r') = \Delta' s \frac{R^{\text{in}}_{lm\omega}(r') R^{\text{up}}_{lm\omega}(r)}{2i\omega A_{\text{in}}} \]
Quasinormal mode response

- Time domain response from inverse Laplace transform

\[
G(x^\mu, x'^\mu) \sim \sum_{l,m} \int d\omega \, e^{-i\omega(t-t')} \Delta'_{s} \frac{R_{l\mu\omega}(r') R_{l\mu\omega}(r)}{2i\omega A_{in}}
\]

\[
G_{QNM} \sim \sum_{lmn} e^{-i\omega_n(t-t')} \frac{\Delta'_{s} R_{lmn}^{in}(r') R_{lmn}^{up}(r)}{\omega_n \partial\omega A_{in}|_{\omega_n}}
\]

\[
A_{in}(\omega) = 0
\]
Ringdown and QNMs

Figure 3 confirms our basic findings from the nonrotating case: the convergence of the QNM expansion is not necessarily monotonic, and the excitation coefficient expansion works better for higher values of $l$. Notice that a relatively small number of overtones is sufficient to reproduce the numerical waveform at early times even when the spin of the Kerr BH is rather large ($j = 0.98$), so that one may in principle expect that a larger number of overtones would be necessary (see e.g. [29, 54–57]). To our knowledge, the calculation presented in this Section is the first concrete proof that an excitation-coefficient expansion is applicable and useful in the Kerr case: all calculations available in the literature so far were specific to the Schwarzschild case (see e.g. [32, 33]).
Late-time tail radiation

- Branch cut associated with power-law decay
- Seen at very late times
- Physically, due to backscatter off long-range potential

\[ G_{BC} \propto \frac{(r_\ast r_\ast')^{l+1}}{(t - t')^{2l+3}} \]

Harms, Bernuzzi, Brugmann (2012)
Part 2

RINGDOWN AND BINARY BLACK HOLES
GWs from compact binaries

- Entire inspiral-merger-ringdown modeled and calibrated to simulations
GWs from compact binaries
First detection
First detection

![Graph showing strain noise versus frequency (Hz)]
First detection

![Graph showing strain noise vs frequency](image)
First detection

\[ 36M_\odot - 29M_\odot \]
Ringdown of GW150914?
The ringdown of GW150914

- Freq and decay of lowest overtone for $\ell = 2, m = 2$
- Consistent with GR $f \propto 30 \left( \frac{M_\odot}{M} \right)$ kHz

LVC arXiv:1602.03841
Consistency tests of GR

- GW150914: Only 1 mode measured
- Consistency test still possible: split signal
- Compare inspiral with merger-ringdown
- Both consistent with IMR analysis

LSC, arXiv:1602.03841
Consistency tests of GR

- Consistency tests can be stacked over many observations
- Ghosh et al. (2016): after ~100 observations at SNR 25, percent level test are achievable

Ghosh et al arXiv:1602.02453
Testing GR with ringdown

- The (2,2) mode dominates
- Large SNR is needed to detect additional modes (~100)
- \[ \rho_{RD} \propto \frac{M_z^{3/2}}{S_n} \]
- Hard for ground-based detectors
- Easier for more massive binaries

---

Berti et al. arXiv:1605.09286
Next Generation Detectors
Space-based detectors

- LISA to fly 2034
- Pathfinder a great success
- Space-based missions open many frontiers: SMBH binaries, WD binaries
- Precision tests of GR with EMRIs

Armano et al. (2016)
Spectroscopy from space

• Space-based BBH detections have huge SNR, $\gtrsim 10^6$
• Ringdown loud enough to measure second mode well
• Many mergers at high redshift
• EMRIs: ringdown at high spin possible

Berti et al. arXiv:1605.09286
Summary

• QNMs outcome of wave eqns on BH spacetimes
• Explored using scalar wave eqs
• QNMs are decaying resonances, for BHs correspondence with null orbits
• QNM spectrum determined by mass and spin
• Rapidly rotating BHs have collective modes (didn’t get to these slides!)