Quasinormal Modes 1: Nonspinning Black Holes

XII Mexican School on Gravity and Mathematical Physics

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Simple model

- Perturbed spacetime

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- Consider Schw BHs:

$$g^S_{\mu\nu} = \text{diag}[-f, f^{-1}, r^2, r^2\sin^2\theta]$$
$$f = 1 - \frac{2M}{r}$$

Scalar wave equation

- Simple toy model: Klein-Gordon eqn

$$\Box_g \Phi = \nabla^\mu \nabla_\mu \Phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu} \sqrt{-g} \partial_\nu \Phi \right) = -4\pi T$$

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$$\Box_g \Phi = -\frac{1}{f} \partial_t^2 \Phi + \frac{1}{r^2} \partial_r (f r^2 \partial_r \Phi) + \frac{1}{r^2} \Delta_{S^2} \Phi$$

$$\Delta_{S^2} = \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta) + \frac{1}{\sin^2\theta} \partial_\phi^2$$

Scalar wave equation

- Separation of variables

$$\Phi = \frac{1}{2\pi} \sum_{\ell m} \int d\omega e^{-i\omega t} \frac{\psi_{\ell m\omega}}{r} Y_{\ell m}(\theta, \phi)$$

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$$\omega^2\psi + f\partial_r(f\partial_r\psi) - \frac{f}{r}(\partial_r f)\psi - \frac{f}{r^2}\ell(\ell+1)\psi = -rfT_{\ell m\omega} = S(r)$$

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- Motivates

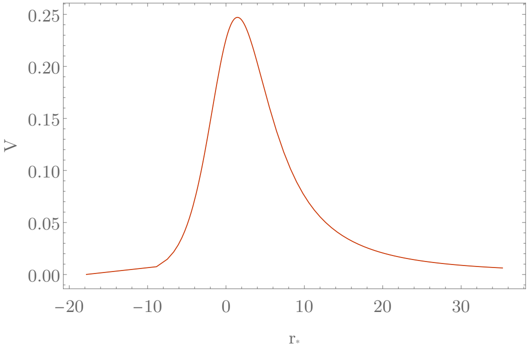
$$\frac{dr_*}{dr} = \frac{1}{f} \qquad r = r + 2M \ln\left(\frac{r}{2M} - 1\right) \qquad r_* \in (-\infty, \infty)$$

Radial equation

- Result is a Schroedinger eqn:

$$\frac{d^2\psi}{dr_*^2} + (\omega^2 - V)\psi = S$$

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]$$



Homogenous solutions

- Far from hole, $\psi'' + \omega^2\psi = 0$

$$\psi \sim e^{\pm i\omega r_*}$$

$$e^{-i\omega t}\psi \sim e^{-i\omega(t\mp r_*)} = \begin{cases} e^{-i\omega u} \\ e^{-i\omega v} \end{cases}$$

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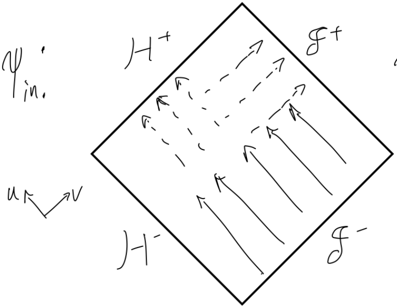
- Slns some linear combination

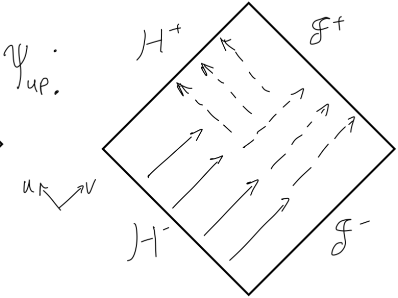
$$\psi_{\text{in}} \sim \begin{cases} A_{\text{out}}(\omega)e^{i\omega r_*} + A_{\text{in}}(\omega)e^{-i\omega r_*}\,, & r_* \to \infty \\ e^{-i\omega r_*}\,, & r_* \to -\infty\,, \end{cases}$$

$$\psi_{\text{up}} \sim \begin{cases} e^{i\omega r_*}\,, & r_* \to \infty\,, \\ B_{\text{up}}(\omega)e^{i\omega r_*} + B_{\text{down}}(\omega)e^{-i\omega r_*}\,, & r_* \to -\infty \end{cases}$$

Scattering states

$$\psi_{\text{in}} \sim \begin{cases} A_{\text{out}} e^{i\omega r_*} + A_{\text{in}} e^{-i\omega r_*} \\ e^{-i\omega r_*} \end{cases} \qquad \psi_{\text{up}} \sim \begin{cases} e^{i\omega r_*} \\ B_{\text{up}}(\omega) e^{i\omega r_*} + B_{\text{down}}(\omega) e^{-i\omega r_*} \end{cases}$$





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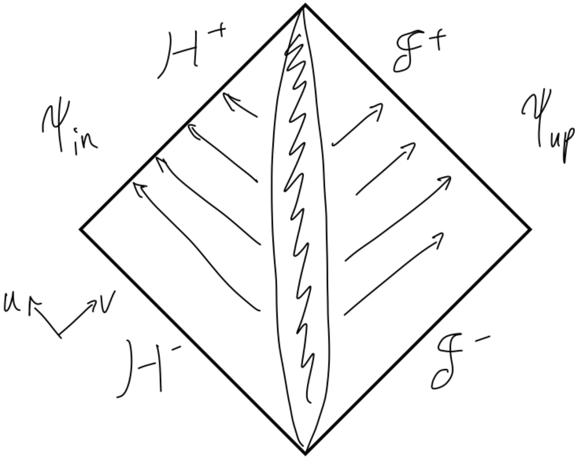

Scattering amplitudes

- Amplitudes give reflection and transmission coeff:

$$\mathcal{T} = \frac{1}{A_{\text{in}}} \qquad \mathcal{R} = \frac{A_{\text{out}}}{A_{\text{in}}}$$

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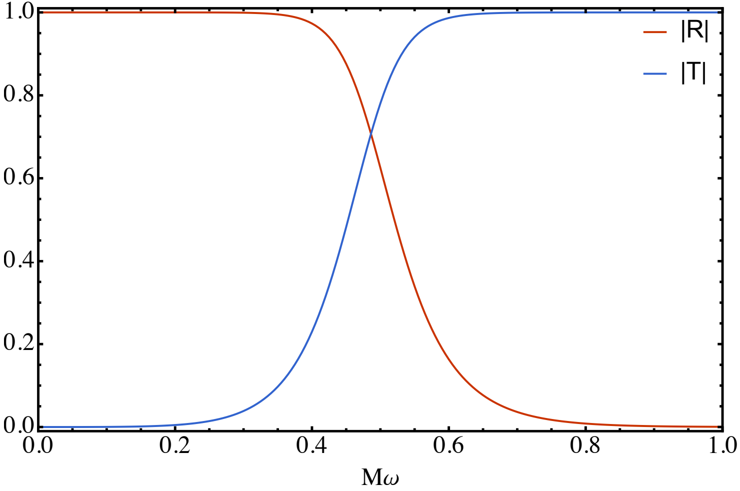
$$W[\psi_{\text{in}}, \psi_{\text{in}}^*] = 2i\omega = 2i\omega(|A_{\text{in}}|^2 - |A_{\text{out}}|^2)$$
$$\Rightarrow 1 + |A_{\text{out}}|^2 = |A_{\text{in}}|^2$$

$$|\mathcal{T}|^2 + |\mathcal{R}|^2 = 1$$

Scattering amplitudes



Scattering amplitudes

- General properties

$$\mathcal{T} = \frac{1}{A_{\text{in}}} \qquad \mathcal{R} = \frac{A_{\text{out}}}{A_{\text{in}}}$$

$$|\mathcal{T}|^2 + |\mathcal{R}|^2 = 1$$

- $\mathcal{R} \to 0$ as $\omega \to \infty$
- $|\mathcal{R} \to 1$ as $\omega \to 0$
- A key feature: for some discrete ω_{lmn}

$$A_{\text{in}}(\omega_{lmn}) = 0$$

Quasinormal modes

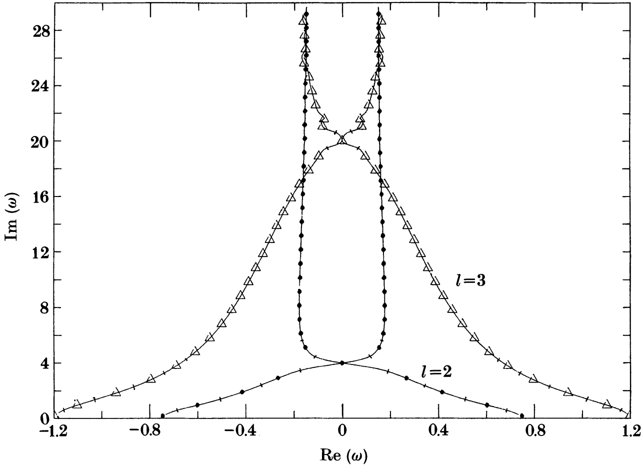


Figure: Leaver 1985

Geometric optics

- To get intuition, take limits. Fast phase:

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- Define $k_\mu = \partial_\mu\alpha$, $\alpha = k_\mu x^\mu$

$$k^\mu k_\mu = 0 \qquad\qquad \nabla_\mu(k^\mu k_\mu) = 0 \Rightarrow k^\mu \nabla_\mu k_\nu = 0$$

Geometric Optics

- Look for null geodesics that don't scatter: unstable bound orbits
- Example: equatorial orbits

$$ds^2 = 0 \Rightarrow -f\left(\frac{dt}{d\phi}\right)^2 + \frac{1}{f}\left(\frac{dr}{d\phi}\right)^2 + r^2$$

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$$b = \frac{\mathcal{L}}{\mathcal{E}} = \frac{r^2 d\phi/d\lambda}{f dt/d\lambda} = \frac{r^2}{f}\frac{d\phi}{dt}$$

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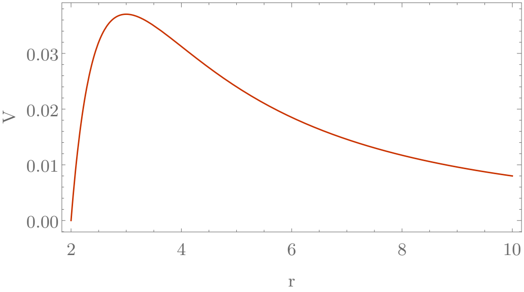
- Get orbit equation

$$\frac{1}{b} = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + V_{\text{eff}}$$

$$\frac{dV_{\text{eff}}}{dr} = 0 \Rightarrow r_0 = 3M$$

Geometric optics

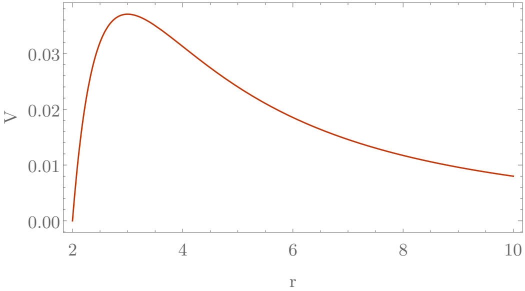


- Can solve

$$b = 3\sqrt{3}M \qquad \Omega = \frac{d\phi}{dt} = b\,\frac{f}{r^2}\bigg|_{r_0} = \frac{1}{3\sqrt{3}M}$$

- Taking into account pattern speed, expect $\omega \approx \ell\Omega$

Geometric optics



- Decay: consider time scale for perturbed orbits to diverge from r_0

$$\frac{1}{b} = \frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + V_{\text{eff}}\big|_{r_0} + \frac{1}{2}\left.\frac{d^2 V_{\text{eff}}}{dr^2}\right|_{r_0}(r - r_0)^2$$

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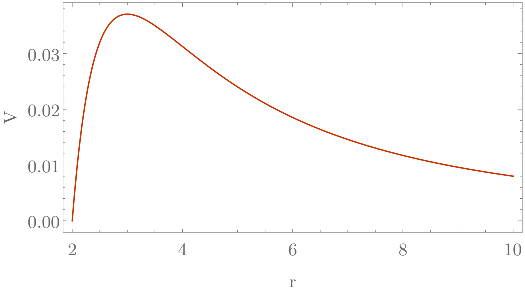

$$\left(\frac{dr}{d\phi}\right)^2 - (r - r_0)^2 \Rightarrow r = r_0 + e^{\phi} = r_0 + e^{\Omega t}$$

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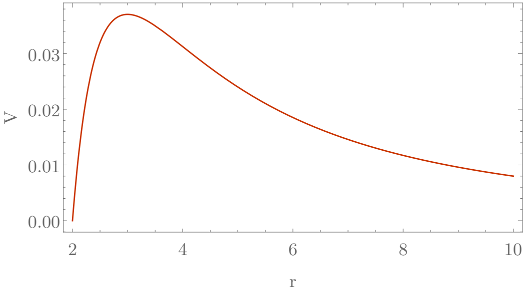

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- Lyapunov exponent:

$$\gamma = \frac{1}{3\sqrt{3}M}$$

WKB approximation

- An equivalent approach is WKB, assuming $M\omega \gg 1$ and $\ell \gg 1$:

$$\psi \sim e^{S_0 + S_1 + \ldots} \qquad S_0 = \pm i \int^{r_*} \sqrt{\omega^2 - V(r)} dr_*$$

- Must match free slns across peak

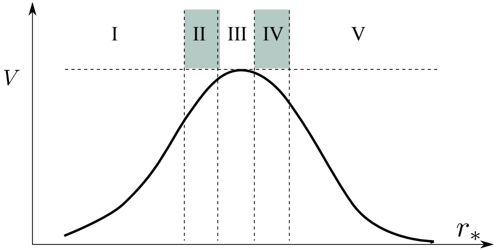


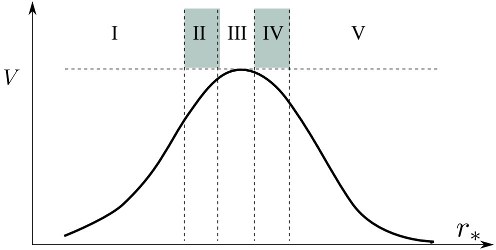
Figure: Yang, Zhang, AZ, Chen (2014)

WKB approximation

- Consistent matching gives

$$\omega_R = (\ell + 1/2)V(r_0) = \frac{\ell + 1/2}{3\sqrt{3}M}$$

$$\omega_I = -\left.\frac{\sqrt{-2d^2V/dr_*^2}}{V}\right|_{r_0} (n + 1/2) = -\frac{n + 1/2}{3\sqrt{3}M}$$



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- Direct integration and "shooting" can get QNMs

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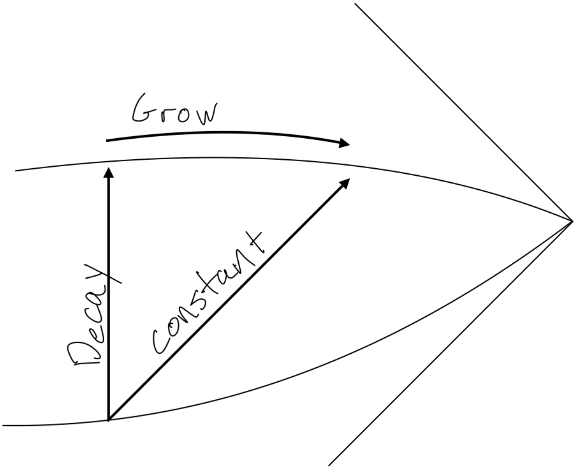
$$\frac{d^2\psi}{dr_*^2} + (\omega^2 - V)\psi = 0$$

QNM blowup towards spatial infinity

$$\psi_{\text{up}} \sim e^{i\omega r_*} = e^{i(\omega_R - i\gamma)r_*} = e^{i\omega_R r_*} e^{\gamma r_*}$$



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$$\frac{d^2\psi}{dr_*^2} + (\omega^2 - V)\psi = 0$$

- Leaver (1985) provides the most accurate and efficient method
- Frobenius sln to ψ, given BCs:

$$\psi = \left(\frac{r}{2M} - 1\right)^{-2iM\omega} e^{i\omega[r-2M+4M\ln(r/2M)]} \sum_{n=0}^{\infty} a_n \left(1 - \frac{2M}{r}\right)^n$$

- For QNMs, sum is convergent

Leaver's Method

- Sub to get a recurrence relation

$$\alpha_0 a_1 + \beta_0 a_0 = 0$$
$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0$$

- $\alpha_n, \beta_n, \gamma_n$ depend on ω, ℓ, n

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- Can iterate to get

$$\frac{a_1}{a_0} = -\frac{\beta_0}{\alpha_0} = -\frac{\gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 - \dots}$$

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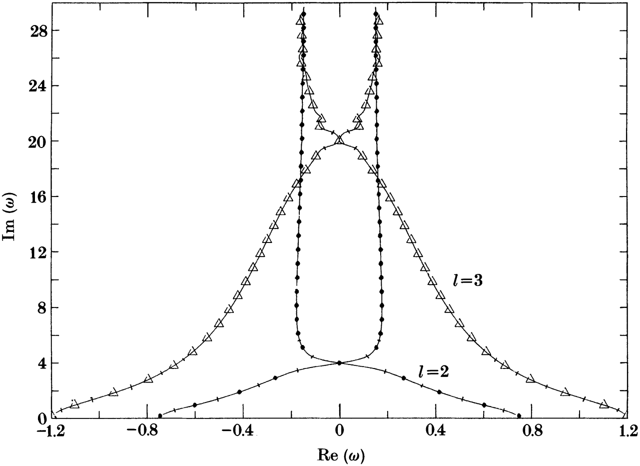
- The CF eventually converges
- Truncate at some N, do a root solve for $\omega_{\ell mn}$
- Also provides QNM wavefunctions $\psi_{\ell mn}$

Quasinormal modes



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QNM excitation

- So far, just hom slns
- Transient sources and ID excite QNM ringing

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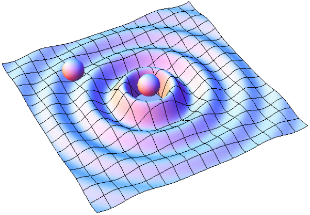
- Green function (propagator):

$$\Box G(x^\mu, x^{\mu\prime}) = -\delta_4(x^\mu, x^{\mu\prime})$$

$$\Phi(x^\mu) = \int d^4x' \sqrt{-g'}\, G(x^\mu, x^{\mu\prime})[4\pi T(x^{\mu\prime})] + (\text{BCs})$$



Green function

- GF also has mode decomp

$$G(x^\mu, x^{\mu\prime}) = \frac{1}{2\pi} \sum_{\ell m} \int d\omega\, e^{-i\omega(t-t')} Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta',\phi') g(r,r')$$

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- Solve by "variation of parameters"

$$g(r, r') = \begin{cases} \dfrac{\psi_{\rm up}(r)\psi_{\rm in}(r')}{2i\omega A_{\rm in}} & r > r' \\ \dfrac{\psi_{\rm up}(r')\psi_{\rm in}(r)}{2i\omega A_{\rm in}} & r < r' \end{cases}$$

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$$\psi_{\text{in}} \sim \begin{cases} A_{\text{out}}(\omega)e^{i\omega r_*} + A_{\text{in}}(\omega)e^{-i\omega r_*}\,, & r_* \to \infty \\ e^{-i\omega r_*}\,, & r_* \to -\infty\,, \end{cases}$$

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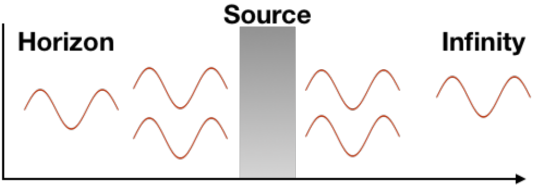

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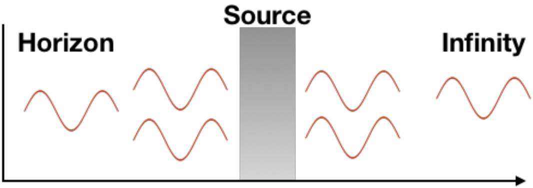
- Normalization from Wronskian of two solutions

$$W[\psi_{\rm in}, \psi_{\rm up}] = 2i\omega A_{\rm in} \qquad (r \to \infty)$$



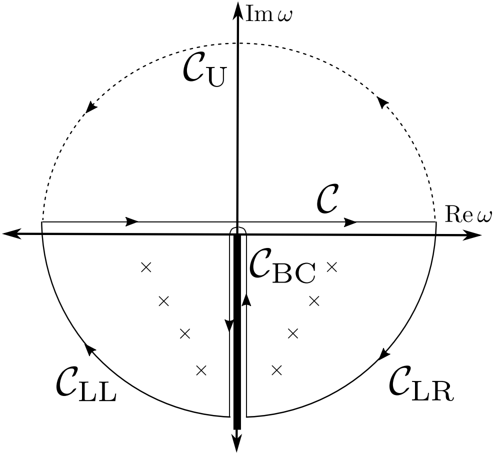
Inverse Laplace transform

$$G(x^\mu, x^{\mu\prime}) = \frac{1}{2\pi}\sum_{\ell m}\int d\omega\, e^{-i\omega(t-t')}Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi')g(r,r')$$

- Integral over source and ID projects onto harmonic basis
- Focusing on an ℓ, m mode with $r > r'$,

$$\begin{aligned}\Phi_{\ell m} &\propto \int_{-\infty+ic}^{\infty+ic} d\omega \frac{e^{-i\omega t}\psi_{\rm up}(r)}{2i\omega A_{\rm in}} \int dr' \psi_{\rm in}(r')S_{\ell m\omega}(r') \\ &= \int_{-\infty+ic}^{\infty+ic} d\omega \frac{e^{-i\omega t}\psi_{\rm up}(r)}{2i\omega A_{\rm in}} f(\omega)\end{aligned}$$

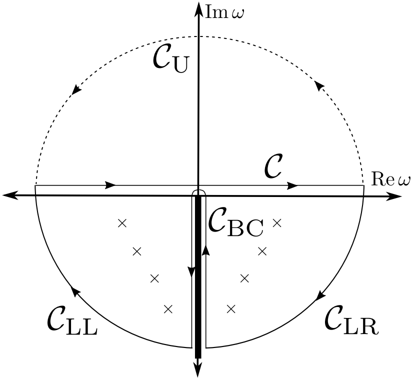
Inverse Laplace transform



$$\Phi_{\ell m} \propto \int d\omega \frac{e^{-i\omega t}\psi_{\mathrm{up}}(r)}{2i\omega A_{\mathrm{in}}} f(\omega)$$

- Slns depend on analytic properties in ω
- $A_{\mathrm{in}} = 0$ are poles
- $\omega = 0$ is a branch point
- Domain of convergence gives causality

Example: far source, far observer



- Let $r > r' \gg M$, source $\delta(r_0)\delta(t_0)$

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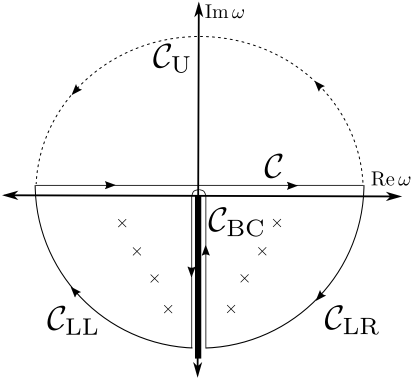
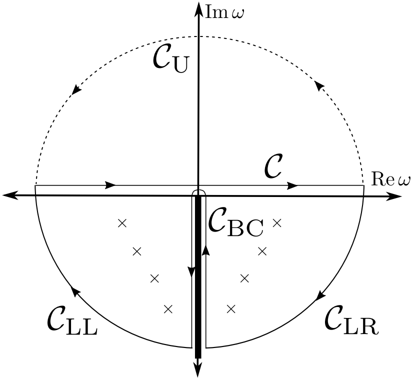


- Let $r > r' \gg M$, source $\delta(r_0)\delta(t_0)$

$$\Phi_{\ell m} \propto \int d\omega \frac{e^{-i\omega(u-u_0)}}{2i\omega} + \int d\omega \frac{A_{\rm out} e^{-i\omega(u-v_0)}}{2i\omega A_{\rm in}}$$

Example: far source, far observer



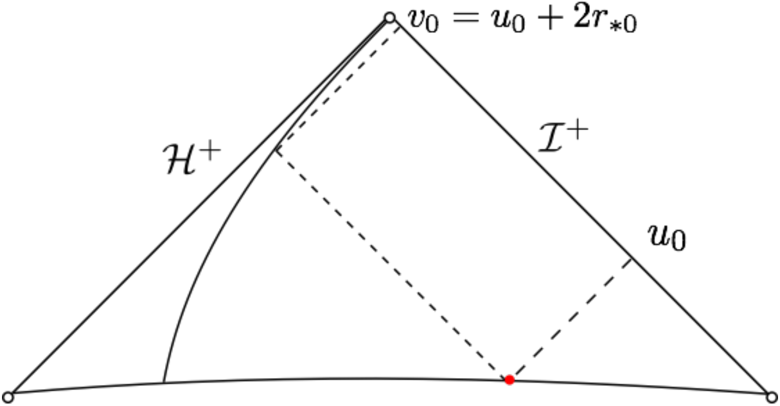
- Let $r > r' \gg M$, source $\delta(r_0)\delta(t_0)$

$$\Phi_{\ell m} \propto \int d\omega \frac{e^{-i\omega(u-u_0)}}{2i\omega} + \int d\omega \frac{A_{\mathrm{out}} e^{-i\omega(u-v_0)}}{2i\omega A_{\mathrm{in}}}$$

- For $u > v_0$ pick up poles by Cauchy's Thm

$$\Phi_{\ell m}^{\mathrm{QNM}} \propto \sum_n \left. \frac{A_{\mathrm{out}}}{\omega \partial_\omega A_{\mathrm{in}}} \right|_{\omega_n} e^{-i\omega_n(u-v_0)}$$

Example: far source, far observer



QNM excitation

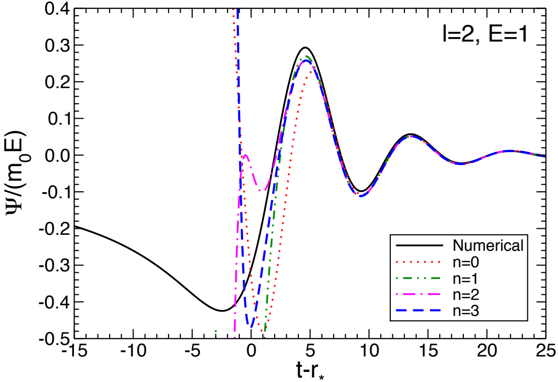


Figure: Zhang, Berti, Cardoso (2013)

Gravitational perturbations

- Perturbations split naturally into odd (magnetic) and even (electric) parities
- Odd (Regge-Wheeler):

$$h_{\mu\nu} = e^{-i\omega t}\begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ * & 0 & 0 & h_1(r) \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \sin\theta[\partial_\theta Y_{\ell 0}(\theta,\phi)]$$

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- One variable suffices

$$\Psi^- = \frac{f}{r} h_1 \qquad\qquad h_0 = \frac{i}{\omega} \frac{d\Psi^-}{dr_*}$$

Gravitational perturbations

- Radial eq nearly the same

$$\frac{d^2\Psi^-}{dr_*^2} + (\omega^2 - V_-)\Psi^- = S$$

$$V_- = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{6M}{r^3}\right]$$

- Asymptotics, scattering states, QNMs largely unchanged

Gravitational perturbations

- The same is true for even parity (Zerilli-Moncrief)

$$h_{\mu\nu} = e^{-i\omega t}\begin{pmatrix} H_0 f & H_1 & 0 & 0 \\ * & H_2/f & 0 & 0 \\ * & * & r^2 K & 0 \\ * & * & * & r^2 K \sin^2\theta \end{pmatrix} Y_{\ell 0}(\theta,\phi)$$

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- H_0, H_1, H_2, K all related to Ψ^+

$$\frac{d^2\Psi^+}{dr_*^2} + (\omega^2 - V_+)\Psi^+ = S$$

- V_+ is more complicated, but same conclusions
- Ψ^- and Ψ^+ are isospectral