Classical and quantum aspects of brane-world cosmology

Rubén Cordero∗ and Efraín Rojas†

∗Departamento de Física, Escuela Superior de Física y Matemáticas del IPN
Unidad Adolfo López Mateos, Edificio 9, 07738, México D.F., México
†Departamento de Física, Facultad de Física e Inteligencia Artificial,
Universidad Veracruzana, 91000 Xalapa, Ver., México

Abstract. We give a brief overview of several models in brane-world cosmology. In particular, we focus on the asymmetric DGP and Regge-Teiltelboim models. We present the associated equations of motion governing the dynamics of the brane and their corresponding Friedmann-like equations. In order to develop the quantum Regge-Teiltelboim type cosmology we construct its Ostrogradski Hamiltonian formalism which naturally leads to the corresponding Wheeler-DeWitt equation. In addition, we comment on possible generalizations for these models including second order derivative geometrical terms.

Keywords: Cosmology, brane-worlds, quantum cosmology
PACS: 04.50.-h, 04.60.Ds, 0.460.Ks, 98.80.Jk

INTRODUCTION

The suggestion that our universe could be a 3 + 1 dimensional surface embedded in a higher dimensional spacetime was proposed by Regge and Teiltelboim (RT) long time ago, motivated by the idea that gravitation can be described in a point- or string-like fashion, as the worldvolume swept out by the motion of a three-dimensional spacelike brane evolving in a higher dimensional bulk spacetime [1]. Some years later Rubakov and Shaposhnikov [2, 3] proposed that our universe could be a topological defect embedded in an extra dimensional background spacetime. The Brane World Scenarios (BWS) [4, 5] is an alternative, besides 4-dimensional general relativity, to understand the birth and then the evolution of our universe. Based in the proposal that our universe can be viewed as a 4-dimensional spacetime object embedded in an N-dimensional background spacetime, (N > 4), the main physical idea behind BWS is that matter fields are confined to a 3-dimensional space (brane) while gravitational fields are present in a higher-dimensional space (bulk), where graviton can travel into the extra dimensions. BWS, besides of resolving the hierarchy problem, have been applied to a plethora of situations such as dark matter/energy and inflation. In the context of cosmology there are predictions of these ideas, that could be tested by astronomical observations. This constitutes one of the several reasons for which BWS are so attractive.

In these BWS, gravity on the brane is recovered by compactifying the extra dimensions [4] or by introducing an Anti de Sitter (AdS) background spacetime [5]. However, Dvali, Gabadadze and Porrati (DGP) [6] showed that, even in an asymptotically Minkowski bulk, 4-dimensional gravity is recovered if a brane curvature term in the
action is included. DGP proposal consider the $Z_2$ reflection symmetry with respect to the brane thus obtaining that gravity, is 4-dimensional on smaller scales $r << r_0$ ($r_0$ is defined in next section), and becoming 5-dimensional on larger distances. It is relevant to note that the reflection symmetry is not the only possibility in these models, for instance, when the brane is coupled to a 4-form field [7] and another antisymmetric cases [8, 9, 10]. In fact, in a pioneer work, Brown and Teitelboim worked out the process of membrane creation by an antisymmetric field motivated in the Schwinger process of pair creation induced by the presence of an electric field [11]. It is worthy to remark that in all these works a brane curvature term is not included in the action. In this paper we write the Friedmann-like equation for a universe described by a brane curvature term interacting with a 4-form field.

At the beginning of the universe, when it was small, the knowledge of quantum effects is relevant to describe the universe in that epoch. Quantum cosmology is intended to describe some properties of the very early universe by means of the Wheeler-DeWitt equation. Brane-world quantum cosmology has several distinctive features with respect to the standard 4-dimensional quantum cosmology. For example, the brane universe has a classically disconnect embryonic epoch [12] besides the boundary conditions are more transparent in brane-world cosmology [13]. These relevant reasons lead to explore alternative technics to study the quantum effects in brane cosmology.

In this paper we present a brief survey of the classical and quantum descriptions of the asymmetric DGP and RT models by using the Ostrogradski Hamiltonian formalism. In addition, we comment on possible generalizations for these models including second order derivative geometrical terms.

The paper is organized as follows. In the next section we give the equations of motion for a brane evolving in a background spacetime. Section 3 is devoted to the Hamiltonian formalism of Regge-Teitelboim set up as a second order derivative theory. The quantization of the model is developed in Section 4. Finally, we give our conclusions.

**CLASSICAL BRANE-WORLD EQUATIONS**

We consider the following action for a brane evolving in a fixed bulk

$$ S_b = S_G^{(4)} + S_m^{(4)} = \frac{M_4^2}{2} \int_b d^4x \sqrt{-g} R + \int_b d^4x \sqrt{-g} L_m. $$

where $M_4$ is the Planck mass and $R$ is the scalar curvature of the trajectory swept out by the brane. Regge and Teitelboim [1] assumed pure gravity $S_G^{(4)}$, however, generalizations have been consider including $S_m^{(4)}$ [12] \(^1\). The simplest form of the matter term is a 4D cosmological constant, $L_m = \rho_v = const$, which corresponds to the brane tension term. The integration in (1) is performed over a (3+1)-dimensional surface describe by $y^A = y^A(x)$ in an $N$-dimensional embedding spacetime and the metric $g_{\mu\nu} = G_{AB} \frac{\partial y^A}{\partial x^\mu} \frac{\partial y^B}{\partial x^\nu}$.

\(^1\) There are other possible generalizations including extrinsic curvature terms in the action, representing a rigidity effect. This is under current investigation.
is the induced metric on the surface where $G_{AB}$ is the bulk metric and $g$ is the determinant of $g_{\mu \nu}$. Here, $y^A \ (A = 0, 1, ..., N - 1)$ are the coordinates in the embedding spacetime and the surface is parametrized by the coordinates $x^\mu \ (\mu = 0, 1, 2, 3)$. The independent dynamical variables of the theory are not the metric components $g_{\mu \nu}$ but the embedding functions $y^A(x)$, with the bulk metric being fixed.

In most of the work on brane cosmology, the bulk geometry is assumed to be dynamical, while the curvature term on the brane is omitted. The general $N$-dimensional action includes the bulk curvature and its matter content

$$S = S_G^{(N)} + S_m^{(N)} = \frac{M_{N}^{N-2}}{2} \int d^N y \sqrt{-G} \mathcal{R} + \int d^N y \sqrt{-G} \mathcal{L}_m,$$

where $M_N$ is the mass Planck in $N$-dimensions. The matter term is usually taken in the form of a bulk cosmological constant, which may take different values on the two sides of the brane. The 4D gravity on the brane is then recovered either by compactifying the extra dimensions [4] or by introducing a large negative cosmological constant in the bulk, which causes the bulk space to warp, confining low-energy gravitons to the brane [5].

Recently, Dvali, Gabadadze and Porrati (DGP) [6] have pointed out that 4D gravity can be recovered even in an asymptotically Minkowski bulk, provided that one includes the brane curvature term. The complete action for this case is

$$S = S_G^{(N)} + S_m^{(N)} + S_G^{(4)} + S_m^{(4)}.$$

Assuming a 5-dimensional bulk and a $Z_2$ symmetry of reflections with respect to the brane, they found that gravity on the brane is effectively 4D on scales $r < < r_0$, with $r_0 = M_5^2 / 2M_4^3$, and becomes 5D on larger scales.

The DGP model can be extended in several directions. A possible extension is to lift the requirement of $Z_2$ symmetry e.g. when the brane is coupled to a 4-form field, so that the 5D cosmological constant is different on the two sides of the brane.

The variation of the action (3) with respect to $G_{AB}$ gives

$$\delta S = -\frac{1}{2k} \int_M d^N y \sqrt{-G} \left( \mathcal{R}^{AB} - \frac{1}{2} G^{AB} \mathcal{R} - k \mathcal{F}_{bulk}^{AB} \right) \delta G_{AB}$$

$$-\frac{1}{2k'} \int_B d^4 x \sqrt{-g} \left( R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} \mathcal{R} - k'T^{\mu \nu} \right) \delta g_{\mu \nu},$$

where

$$\mathcal{F}_{bulk}^{AB} = \frac{2}{\sqrt{-G}} \frac{\delta L_m}{\delta G_{AB}}, \quad T^{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta L_m}{\delta g_{\mu \nu}},$$

are the bulk and the brane energy-momentum tensors, respectively, $k = M_5^{2-N}, k' = M_4^{2-N}$, and the variation $\delta g_{\mu \nu}$ is to be expressed in terms of $\delta G_{AB}$ by means of $g_{\mu \nu} = \int_M d^N y G_{AB}(y) \partial_\mu y^A(x) \partial_\nu y^B(x) \delta^N(y^A - y^A(x))$. We thus obtain the $N$-dimensional Einstein’s equations

$$\mathcal{R}^{AB} - \frac{1}{2} G^{AB} \mathcal{R} = k(\mathcal{F}_{bulk}^{AB} + T_{brane}^{AB}),$$

where $\mathcal{F}_{bulk}^{AB}$ and $T_{brane}^{AB}$ are the bulk and brane energy-momentum tensors, respectively, and $k$ is the cosmological constant.
where

\[ T_{\text{brane}}^{AB} = \frac{1}{\sqrt{-G}} \int d^4x \sqrt{-g} \left[ T^{\mu\nu} - \frac{1}{k'} \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \right] \partial_\mu y^A(x) \partial_\nu y^B(x) \delta^N (y^A - y^A(x)) . \]  

(7)

From Eqs. (6) and (7) we see immediately that if 4D Einstein’s equations are satisfied on the brane, \( \tilde{T}^{\mu\nu} \equiv T^{\mu\nu} - \frac{1}{k'} (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0 \) hence the brane has no gravitational effect on the N-dimensional bulk. The brane can then be treated as evolving in a fixed background geometry. We name such branes as “stealth” branes [13].

The brane equations of motion can be obtained by varying the action with respect to the embedding functions \( y^A(x) \). These equations can be expressed as [14]

\[ \tilde{T}^{\mu\nu} K^{\mu\nu}_i = 0 \]  

(8)

where \( K^{\mu\nu}_i = -n_i^A D_\mu e^A_\nu \) is the extrinsic curvature, \( n^i \) are the (N-4) unit normal vectors to the worldsheet with tangent vectors \( e^A_\mu = y^A_{,\mu} \), and \( D_\mu = e^A_\mu \nabla_A \), with \( \nabla_A \) being the covariant derivative in the metric \( G_{AB} \).

Suppose that we have a solution of 4D Einstein’s equations (\( G^{\mu\nu} = k'T^{\mu\nu} \)) which can be embedded into a N-dimensional bulk with some functions \( y^A(x) \). It is then clear that such an embedding gives a solution both of the N-dimensional Einstein’s equations (6) and of the brane equations of motions (8). Thus any 4D Universe which is embeddable into the bulk represents a possible internal evolution of a brane world. However a stealth brane is not our universe [13].

We now turn to a more general situation, when 4D Einstein’s equations on the brane are not necessarily satisfied. Here, we consider the case of spherical branes in a 5D bulk for simplicity. We assume a non-zero bulk cosmological constant and allow for the possibility that it can take different values on the two sides of the brane.

Following the same steps as in [15], the equations of motion of the brane can be expressed as

\[ [K] g_{\mu\nu} - [K_{\mu\nu}] = k \tilde{T}_\mu^{\nu} , \]  

(9)

\[ \tilde{T}^{\mu\nu} < K_{\mu\nu} > = \left[ \mathcal{T}_n \right] , \]  

(10)

\[ \nabla_\nu (T_\mu^\nu) = -\left[ \mathcal{T}_\mu \right] . \]  

(11)

Here, \( \mathcal{T}_n = (\mathcal{T}_{\text{bulk}})_{AB} n^A n^B \), \( \mathcal{T}_{\mu\nu} = (\mathcal{T}_{\text{bulk}})_{AB} e^A_\mu e^B_\nu \), the square and angular brackets stand for the difference and the average of the corresponding quantity on the two sides of the brane respectively, e.g., \( [K_{\mu\nu}] = K^{+}_{\mu\nu} - K^{-}_{\mu\nu} \), and \( < K_{\mu\nu} > = \frac{1}{2} (K^{+}_{\mu\nu} + K^{-}_{\mu\nu}) \), where “+” and “−” correspond to the brane exterior and interior, respectively.

We take the following form for the bulk energy-momentum tensor, \( \mathcal{T}_{\text{bulk}}^{\pm} = -k^{-1} \Lambda^{\pm} g_{AB} \). Then, by using the generalized Birkhoff theorem, the 5D metric can be expressed as

\[ ds^2_5 = -A_\pm d\tau^2 + A_\pm^{-1} da^2 + a^2 d\Omega_3^2 \]  

(12)

where \( d\Omega_3^2 \) is the metric on a unit 3-sphere, \( A_\pm = 1 - \frac{\Lambda^{\pm}}{6} a^2 - \frac{2\mathcal{M}^{\pm}}{M_7^3 a^2} \), and \( \mathcal{M}^{\pm} \) could be interpreted as black holes masses on the two sides of the brane. The embedding function
\( y^A = Y^A(x^\mu) = (t(\tau), a(\tau), \chi, \theta, \phi) \), gives a parametric representation of the trajectory of the brane. The function \( a(\tau) \) is known as the scale factor. In the proper time gauge, the metric on the brane is \( ds_3^2 = -d\tau^2 + a^2(\tau)d\Omega_3^2 \). The non-vanishing components of the extrinsic brane curvature of the worldvolume are

\[
K_{\tau\tau}^\pm = -\frac{\left( \ddot{a} + \frac{1}{2} \frac{\partial A_\pm}{\partial a} \right)}{a^2 + A_\pm}^{1/2}, \quad K_X^\pm = K^\pm_\theta = K^\pm_\phi = \frac{(\dot{a}^2 + A_\pm)^{1/2}}{a}. \quad (13)
\]

Inserting this information into the junction condition, Eqs. (9), and assuming that the brane energy-momentum tensor is of the form \( T^\nu_\mu = \text{diag}(\rho, -P, -P, -P) \), we obtain

\[
(a^2 + A_-)^{1/2} - (a^2 + A_+)^{1/2} = \frac{ka}{3} \left( \rho - \frac{3(\dot{a}^2 + 1)}{k'a^2} \right), \quad (14)
\]

\[
\dot{\rho} + \frac{3}{a} (\rho + P) = 0. \quad (15)
\]

The Eq. (15) represents the energy-momentum conservation on the brane. The former system was discussed in [13] where several interesting cases were treated. A stealth brane corresponds to \( \mathcal{M}^+ = 0 \) and \( \Lambda^+ = \Lambda^- \equiv \Lambda \). As expected, in this case Eqs. (14), (15) reduce to the standard FRW evolution equations. Even if the bulk cosmological constant is non-zero, it has no effect on the evolution of the brane. Suppose now \( \mathcal{M}^- = 0, \rho = \text{const} \). Then, Eq. (14) can be cast as

\[
\left( \frac{a^2 + 1}{a^2 - \Lambda^- / 6} \right)^{1/2} - \left( \frac{\dot{a}^2 + 1}{a^2} - \frac{\Lambda^-}{6} - \frac{\alpha}{6M_5^3} - \frac{2M_4}{M_5^3} a^4 \right)^{1/2} = \frac{1}{M_4^3} \left( \frac{\rho}{3} - M_4^2 a^2 \right), \quad (16)
\]

where we have defined \( \alpha = (\Lambda^+ - \Lambda^-)M_5^3 \). Taking into account the following limits, \( \Lambda^+ , \Lambda^- \rightarrow \Lambda \) and \( M_5 \rightarrow \infty \), in such a way that \( \alpha \) results in a non-null constant, we can expand the second term on the LHS of (16) to get

\[
\left( \frac{\rho}{3} - M_4^2 a^2 \right) \left( \frac{\dot{a}^2 + 1}{a^2} - \frac{\Lambda^-}{6} \right)^{1/2} = \frac{\alpha}{12} + \frac{\mathcal{M}}{a^2}. \quad (17)
\]

The Friedmann-like equation from the former relation is

\[
\dot{a}^2 + \frac{1}{a^2} = \frac{\rho}{3M_4^2} \equiv \mathcal{R} \equiv \ddot{\rho} Y, \quad (18)
\]

where \( Y \) is defined through the equation \( M_4^4 (1 - Y)^2 \left( \frac{\dot{Y}}{\dot{\rho}} \right) = \ddot{\rho}^{-3} \left( \frac{\alpha}{12} + \frac{\mathcal{M}}{a^2} \right)^2 \). At classical level this equation is obtained from the following action modeling a brane interacting with a 4-form field and propagating in a fixed background spacetime with a cosmological constant \( \Lambda \) [16]

\[
S = \frac{M_4^2}{2} \int d^4x \sqrt{-g} (R - \Lambda) + \frac{k_2}{4!} \int d^4x \sqrt{-g} A_{ABCD} \epsilon^{ABCD}, \quad (19)
\]
where $\Lambda_b = 2\rho/M_{(4)}^2$ being the cosmological constant on the brane; $A_{ABCD}$ is a gauge 4-form Ramond-Ramond field onto the bulk; $\varepsilon^{ABCD}$ is the Levi-Civita tensor in the bulk and $k_2$ is the coupling constant between the brane and the antisymmetric tensor.

**OSTROGRADSKI HAMILTONIAN APPROACH**

Now, we construct the Hamiltonian formalism for the RT model with brane tension term. It is convenient at this point to take the following form the metric induced on the worldvolume $ds^2 = -N^2 d\tau^2 + a^2 d\Omega_3^2$. where $N = \sqrt{t^2 - a^2}$, which coincides with the lapse function when we perform an ADM decomposition of the action (1). The Lagrangian of the RT model including cosmological constant once we have integrated out the angular part of the density lagrangian becomes

$$L(i, i, a, \dot{a}, \ddot{a}) = \frac{a^3}{N^2} (a\ddot{a} - a\dot{\dot{a}} + N^2 i) - N a^3 H^2,$$  \hspace{1cm} (20)

where we have introduced the quantity $\tilde{H}^2 = \Lambda/3M_{(4)}^2$. We should remark that the RT model is a genuine second order derivative theory but it is possible to identify a total derivative term in the Lagrangian and, as a consequence, it can be transformed into an effective first derivative theory. However, as was shown in [17], there are several advantages in considering the RT model as a second order derivative model. The RT action specialized to spherical configurations, in terms of an arbitrary parameter $\tau$, is $S_{RT} = 6\pi^2 \int d\tau L(a, \dot{a}, \ddot{a}, i, i)$. A deeper insight of the phase space structure of the theory defined by the Lagrangian (20) is achieved by an Ostrogradski procedure for higher order derivative systems. The highest conjugate momenta to the velocities $\{i, \dot{a}\}$ are, respectively,

$$P_i = \frac{\partial L}{\partial \dot{i}} = -\frac{a^2}{N^3} \dot{a}, \hspace{1cm} P_a = \frac{\partial L}{\partial \dot{a}} = \frac{a^2 i^2}{N^3},$$  \hspace{1cm} (21)

such that the highest momentum spacetime vector is $P_\mu = \frac{a^2}{N^3} (\dot{a}, i, 0, 0, 0)$. Note that the moment is directed normal to the worldvolume.

The conjugate momenta to the position variables $\{i, a\}$ are, respectively

$$p_i = \frac{\partial L}{\partial \dot{i}} = \frac{a^2}{N^3} \dot{a}, \hspace{1cm} p_a = \frac{\partial L}{\partial \dot{a}} = -\frac{a^2}{N^3} \ddot{a} + a^{-2} N^2 V(a) = \left(\frac{\dot{a}}{i}\right) \Omega,$$  \hspace{1cm} (22)

where we have introduced the standard potential in a minisuperspace approach, $V(a) = a^2 (1 - a^2 \tilde{H}^2)$. It is important to mention that the momentum $p_a$ is obtained by two contributions: one coming from the ordinary theory and the other by a surface term, i.e., $p_a = p_a + p_a$, with

$$p_a = -\frac{a^2 i}{N^3} \left[\dot{a}^2 + 2N^2 + a^{-2} N^2 V(a)\right]$$  \hspace{1cm} (24)
\[ p_a = \frac{2a\dot{a}}{N}, \quad (25) \]

where \( p_a \) is the momentum conjugated to \( a(\tau) \) when considering as the Lagrangian \( L_s = \frac{d}{d\tau}(a^2 \dot{a}/N) \).

The appropriate phase space of the system, \( \Gamma = \{ t, a, i, \dot{a}; p_t, p_a, P_t, P_a \} \), is identified explicitly. The Ostrogradski formalism yields the canonical Hamiltonian \( H_0 = p \cdot X + P \cdot \dot{X} - L = p_a \dot{a} + p_t i + J_{\zeta} \), where we have defined \( J_{\zeta} = -\frac{\dot{N}}{N} [a^2 + a^{-2}N^2 V(a)] = \frac{N^2}{T} \Omega \). This potential-like term results in an implicit function of the phase space variables in the combination \( N^3P^2 \). Apparently this may look like an unnecessary complication to write both the physical momentum and \( H_0 \) in terms of \( \Omega \) but this quantity results important because it is nothing but the conserved bulk energy. Indeed, squaring the energy equation (22), results in the evolution equation
\[ N^2 + \dot{a}^2 = \gamma N^2 a^2 \dot{H}^2, \quad (26) \]
where \( \gamma = \gamma(a) \) satisfies the cubic equation \( \gamma(\gamma - 1)^2 = \Omega^2 / a^8 \dot{H}^6 \).

**Constraint analysis**

Since the Lagrangian (20) is linear in the accelerations, we have two primary constraints
\[ \mathcal{C}_1 = P \cdot \dot{X} = 0, \quad (27) \]
\[ \mathcal{C}_2 = NP \cdot n - \frac{a^2 i}{N} = 0. \quad (28) \]

In fact, this is a generic feature of any brane Lagrangian linear in the accelerations possessing reparametrisation invariance. Therefore, the Hamiltonian that generate time evolution of the fields is
\[ H = H_0 + \lambda_1 \mathcal{C}_1 + \lambda_2 \mathcal{C}_2, \quad (29) \]
where \( \lambda_1 \) and \( \lambda_2 \) are Lagrange multipliers enforcing the primary constraints. For any canonical variable \( z \in \Gamma \), we have \( \dot{z} = \{ z, H \} \), on the constraint surface, where the corresponding Poisson bracket of any two functions \( F(z) \) and \( G(z) \) in \( \Gamma \) is defined as
\[ \{ F, G \} = \frac{\partial F}{\partial t} p_t + \frac{\partial F}{\partial a} p_a + \frac{\partial F}{\partial i} P_t + \frac{\partial F}{\partial \dot{a}} P_a - (F \longleftrightarrow G). \quad (30) \]

With this symplectic structure, the constraints (27) and (28) are in involution, \( \{ \mathcal{C}_1, \mathcal{C}_2 \} = 0 \). According to the Dirac-Bergmann program for constrained systems, \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) must be preserved by the evolution which demands the existence of the secondary constraints
\[ \mathcal{C}_3 = H_0 = p \cdot \dot{X} + N(a^2 \dot{H}^2 - \frac{1}{a^3}N^2 P^2) = 0. \quad (31) \]
\[ \mathcal{C}_4 = p \cdot n = 0. \quad (32) \]
The vanishing of the canonical Hamiltonian is expected due to the reparametrization invariance of the RT model and it generates diffeomorphisms normal to the worldvolume. The secondary constraint (32) is characteristic of every linear brane model in accelerations. There are no other constraints. In fact, we have a theory with first- and second-class constraints which is a consequence of the gauge symmetry.

All the constraints should be divided into first- and second-class ones. For the RT model we have two first-class phase space constraints

\[ F_1 = C_1, \]

\[ F_2 = \frac{N^3}{a^2} \left( \frac{3i^2 - N^2 a^2 H^2}{i^2 - 3N^2 a^2 H^2} \right) C_3 + C_4, \]

and two second-class constraints. For our case, the selection of the second-class constraints is straightforward. We choose them as

\[ S_1 = C_2; \]

\[ S_2 = C_4. \]

Thus, we have the presence of 2 gauge transformations in the RT model. The counting of degrees of freedom is as follows: \( dof = [\text{Total number of canonical variables} - 2 \times \text{first-class constraints} - \text{Number of second-class constraints}] / 2 = (8 - 2 \times 2 - 2) / 2 = 1 \) which agrees with the scale factor \( a \).

**BRANE-WORLD QUANTUM COSMOLOGY**

In this section we study the canonical quantization of RT model. We start by promoting the classical constraints into operators, densely defined on a common domain in a proper Hilbert space. As it is well known, we can only achieve a consistent classical theory by implementation of the Dirac bracket. Once this is done, the second-class constraints are eliminated off the theory by converting them into strong identities. At the quantum level this is performed by defining the quantum commutator of two quantum operators as

\[ [\hat{A}, \hat{B}] := i \{A, B\}^*, \]

where the Dirac bracket \( \{\cdot, \cdot\}^* \) is defined as usual. We work from now on in natural units. Thus, with this prescription the operators corresponding to second-class constraints are also enforced as operator identities [18]. For our system, this yields the quantum operator expressions

\[ \hat{\mathcal{F}}_1 = \hat{\Pi}_v - \frac{a^2 i}{N} = 0, \]

\[ \hat{\mathcal{F}}_2 = \hat{p}_a - \frac{2aa}{N} = 0, \]

which, in particular, tell us the character of the quantum operators \( \hat{\Pi}_v \) and \( \hat{p}_a \). We choose to work on the “position” representation, where we consider the position operators by multiplication and their associated momenta operators by \( -i \) times the corresponding derivative operator when applied on states defined on a suitable Hilbert space. By defining the quantum first-class constraints as

\[ \hat{\mathcal{F}}_1 := -iN \frac{\partial}{\partial N}, \]
we will work on the assumption that the commutators of these quantum constraints form a closed Lie algebra. First, we explore the Wheeler–DeWitt equation emerging by considering the physical states $\Psi$ of the theory as those defined by Dirac conditions

$$\hat{F}_1 \Psi = 0 \quad (40)$$
$$\hat{F}_2 \Psi = 0 \quad (41)$$

Eq. (40) establishes that our physical states $\Psi$ are not explicitly depending on the phase space variable $N$. Nevertheless, due mainly to the complexity of our WDW equation (41), we have not succeed in finding explicit solutions for the physically admissible quantum states. We note that the last term in the operator (39) does not contribute to the WDW equation. We propose a stationary state $\Psi(a,t) := e^{-i\Omega t} \psi$, where $\psi := \psi(a)$ satisfies the WDW equation

$$\left[ -\frac{\partial^2}{\partial a^2} + U(a) \right] \psi(a) = 0 \quad (42)$$

where the potential $U(a)$ is given by

$$U(a) = a^2 \left[ (\gamma - 1)H^2 a^2 + 2 \right] \left( 1 - \gamma H^2 a^2 \right) \quad (43)$$

which is recognized as the potential function found in [19] by repeatedly use of the evolution constraint (26) in equation (41). In that work a different theoretical approach was used to the presented here. The behavior of this potential is drawn in Fig. 1, where we can see the characteristic potential barrier. There is a distinctive feature of this potential with respect to the standard 4-dimensional quantum cosmology. It is possible the existence of the classical universe before the quantum tunneling [12]. It can be shown that after considering appropriate boundary conditions the big–bang singularity in our quantum theory can be neutralized by properly choosing the origin as inaccessible to wavepackets. For further details on the behavior of the potential $U(a)$, the reader is referred to [19].
CONCLUSIONS

In this paper we have presented some classical and quantum relevant features of brane cosmology. The equations of motion for the brane universe generalizing the RT model and the DGP brane cosmology were written. In the case of the asymmetric DGP model, we have considered the model of a brane universe interacting with a Ramond-Ramond field. Using the Ostrogradski approach for theories of higher derivatives we have constructed the Hamiltonian formalism for the RT brane cosmology and with the help of Dirac formalism for constrained systems we have obtained the Wheeler-DeWitt equation. A outstanding characteristic of brane-world quantum cosmology is the existence of a classically disconnect embryonic epoch for the universe. Furthermore, the boundary conditions for the Wheeler-DeWitt equations are more transparent in brane-world quantum cosmology than standard 4D quantum cosmology. A possible extension to the RT model is to consider an extrinsic curvature term in the action representing a rigidity effect. This type of model is a genuine second order derivative theory and the Ostrogradski Hamiltonian formalism can be applied to it. This study is under current investigation.

ACKNOWLEDGMENTS

We acknowledge partial support from SNI-México and CONACyT research grant J1-60621-I. ER acknowledge partial support from PROMEP (Mexico). R. C. was partially supported by COFAA-IPN and by SIP-IPN grants 20111070.

REFERENCES

1. T. Regge and C. Teitelboim, in Proc. Marcel Grossman, p. 77 (Trieste, 1975);
3. V. A. Rubakov, Large and Infinite Extra Dimensions, hep-th/0104152;
13. R. Cordero and A. Vilenkin, Phys. Rev. D 65 083519 (2002);