Equation of State of a Self-gravitating Fluid with Rotating Particles

L. Escamilla, José Torres-Arenas and M. Sabido

División de Ciencias e Ingenierías Campus León, Universidad de Guanajuato, Lomas del Bosque
103, León Guanajuato; México

Abstract. The equation of state for a system interacting via a gravitational potential, is obtained. This potential is found in the context of general relativity from the Kerr metric, when considering the so-called weak-field limit and performing an angular average to have a potential with spherical symmetry. The resulting potential is naturally regular and bounded by effect of angular momentum. Microcanonical and canonical ensembles are studied, calculating their thermodynamic potential, finding the corresponding equation of state. Afterwards both ensembles are compared in the diluted regime showing explicitly their equivalence.

Keywords: Self-gravitating systems, Statistical Mechanics
PACS: 04.40.-b, 05.20.-y, 05.70.Ce

INTRODUCTION

Systems interacting via so-called long-range interactions are present in a wide range of areas in physics. However, their statistical and dynamical description is poorly understood in comparison with short-range systems [1]. These systems are present in astrophysics [2, 3, 4], plasma physics [5], atomic physics [7], hydrodynamics [8], etc. The existence of a wide variety of systems with long-range interactions, justify the need to understand the problems present studying systems with long-range interactions. The main concern is the construction of a thermodynamical treatment of those systems and the understanding of analogies and differences between the different domains of long-range systems. We start by defining the concept of short/long-range [1], considering systems of pairwise additive and two-body translationally invariant potentials. For particles interacting each other via a potential that decays at large distances like \( \frac{1}{r^\alpha} \) in a \( d \)-dimensional space, the interaction is consider to be long-range if \( \alpha \leq d \); since the energy \( \varepsilon \), of a given particle at the center of sphere of radius \( R \) where the other particles are homogeneously distributed (excluding the contribution to \( \varepsilon \) from the particles of a small region in the neighborhood of radius \( \delta \), related only to avoid any possible divergence of point particles and it is not related with the long-range nature), is given by:

\[
\varepsilon = \int_\delta^R d^d \rho \gamma \frac{\gamma}{r^\alpha} = \rho \gamma \Omega_d \int_\delta^R r^{d-1-\alpha} dr = \rho \gamma \Omega_d \frac{[R^{d-\alpha} - \delta^{d-\alpha}]}{d - \alpha} \quad \text{if} \quad \alpha \neq d. \tag{1}
\]

where \( \rho \) is the generic density, \( J \) is a coupling constant and \( \Omega_d \) is the angular volume in the \( d \)-dimensional space. When increasing \( R \), \( \varepsilon \) remains finite only for \( \alpha < d \); therefore, the total energy \( E \) grows linearly with the volume \( V \), i.e., the system is extensive.
Such interactions are short-range. For $\alpha \leq d$ the energy $\varepsilon$ diverges, these interactions are called long-range or non-integrable. A generic feature of these systems is the lack of additivity [1] related to the interaction between elements in different subsystems. Linked to non-additivity is another feature present in long-range systems, the non-extensivity of the thermodynamic quantities [6], i.e., parameters which usually are proportional to $N$, do not scale with the size of the system. Another important feature of long-range systems is that ensembles can be inequivalent [9, 10, 11, 12], therefore the study of the statistical mechanics in those systems requires a careful examination of the conditions that determine their equivalence. Perhaps the most peculiar behavior of long-range systems is the possibility to display negative specific heat in the microcanonical ensemble [9, 13, 14]. Microcanonical equilibrium contains all the information of canonical equilibrium, but in case of ensemble inequivalence, the converse is wrong [15].

**THEORY**

When weak-field limit is applied to the Kerr metric [16], which describes the gravitational field produced by a spherical rotating body with mass $M$ and angular momentum $J$, the result is a potential with both radial and angular dependence, i.e. $\phi = \phi(r, \theta)$; an angular average is conducted in order to obtain a spherical symmetric potential. Thus, the effective gravitational potential $U(r) = \langle \phi(r, \theta) \rangle_\theta$ is:

$$U(r) = -\frac{GMa}{r} \arctan \left( \frac{a}{r} \right),$$  

being $a = J/Mc$ the specific angular momentum and $G$ the gravitational constant. This potential is purely attractive, moreover is indeed regular at short distances, since for $r \to 0$, $\arctan(a/r) \to \pi/2$; retaining its long-range nature. Specific angular momentum $a$, is a parameter which modulate the intensity of the gravitational attraction exerted by particles. However, the long-range behavior is the same regardless the value of angular momentum. The microcanonical and canonical ensembles are applied to a system of $N$ point particles interacting via the potential in (2), i.e. a self-gravitating system. This application is inspired in [17], with interaction written as,

$$U(\vec{q}_1, \ldots, \vec{q}_N) = -\frac{GM^2}{a} \sum_{1 \leq l < j \leq N} \arctan \left( \frac{a}{|\vec{q}_l - \vec{q}_j|} \right).$$  

The self-gravitating fluid is conformed by $N$ identical point particles with mass $M$ and angular momentum $a$, contained inside a cubic box of side $L$ and volume $V$. The total energy of the system is $E$ and the particles are interacting gravitationally with each other via Eq. (3).

*Microcanonical Ensemble:* To proceed with the microcanonical ensemble is necessary to find entropy $S(E,V,N)$, through the number of microstates $\Omega(E,V,N)$ via the Boltzmann’s relation $S = k_B \ln \Omega$, with $k_B$ the Boltzmann constant. The movement of the particles in the container is limited between 0 and $L$ in each coordinate; thus integration limits for coordinates in phase space for $\Omega(E,V,N)$ actually are dependent
of the container’s size, and can be rescaled by considering dimensionless variables, $\vec{r}_l = (x_l, y_l, z_l)$, with $1 \leq l \leq N$; which are related to $q_l$ through the change of variable given by: $q_l = L r_l \Rightarrow d^3 q_l = L^3 d^3 r_l$; with $0 \leq (x_l, y_l, z_l) \leq 1$. The integrand of the coordinate partition function is written in terms of two new dimensionless variables obtaining also an effective potential energy per particle $u(\vec{r}_1, \ldots, \vec{r}_N)$; those variables are defined as: $\lambda = EJ/\hbar^2 cN^2$, $\alpha = J/MLc = a/L$. After integrating over the momenta, the entropy is given as

$$S = k_B \ln \left[ \frac{V^N G^{3N/2 - 1} M^{6N - 3} e^{3N/2 - 1} N^{3N - 2}}{N! (2\pi \hbar^2)^{3N/2} J^{3N/2 - 1} \Gamma(3N/2)} \right] + k_B \ln \omega(\lambda, N),$$  \hspace{1cm} (4)

where $\Gamma(3N/2 + 1)$ is the Gamma function, $\hbar$ Planck’s constant divided by $2\pi$ and $\omega(\lambda, N)$ is the coordinate partition function. From (4) is possible to calculate the temperature $T$, and pressure $P$, using the standard thermodynamic relations $1/T = (\partial E/S)_V$ and $P/T = (\partial V/S)_T$. Carrying out the differentiation of entropy with respect of $V$ results in the microcanonical equation of state, $PV/Nk_BT = 1 + (L/3N) \partial V \ln \omega(\lambda, N)$; which has a corrective term due to the interaction between particles. When considering a very weak interaction, $u(\vec{r}_1, \ldots, \vec{r}_N) \rightarrow 0$, the microcanonical coordinate partition function behaves like $\omega(\lambda, N) \rightarrow \lambda^{3N/2 - 1}$; then we have, $\partial V \ln \omega(\lambda, N) \rightarrow 0$; thus, in the limit of weak interaction and $N \rightarrow \infty$, we recover the energy and the equation of state of an ideal gas, $E/Nk_BT = 3/2$ and $PV/NkB = 1$. We define a function of $\lambda$ and $V$, in order to write the thermodynamic quantities as a function of it, $g(\lambda, V) = 1 + L/3N \partial V \ln \omega(\lambda, N)$, then the equation of state can be written using this function, as:

$$\frac{PV}{NkB} = g(\lambda, V).$$  \hspace{1cm} (5)

Thermodynamic quantities like energy, entropy, specific heat and others, can be expressed using the function $g(\lambda, N)$. It is desirable to find an explicit expression for the equation of state. With this purpose the dilute regime is considered. In this regime of very low density, $\lambda \gg 1$, i.e., taking $N \rightarrow 0$ we obtain the thermodynamic quantities as a series in powers of $1/\lambda$, a virial expansion. For simplicity, consider that potential energy per particle, $u$, between any pair of particles is identical to the interaction between particle 1 and particle 2, then the coordinate partition function for second order in $1/\lambda$ is obtained. To this approximation of $\omega(\lambda, N)$ the dilute thermodynamic limit, $N \rightarrow \infty$ while $N/L$ (or $\lambda$) remains constant, is also applied; then the equation of state can be written as,

$$\frac{PV}{NkB} = 1 + \frac{3L}{2\lambda} \frac{\partial}{\partial L} b_0 + \frac{3L}{8\lambda^2} \frac{\partial}{\partial L} [b_1 - 42 b_0^2] + \mathcal{O}(\lambda^{-3}),$$  \hspace{1cm} (6)

with the first virial coefficients $b_0$, $b_1$, $b_2$, which can be easily calculated. Thus, we obtained an analytical expression for the microcanonical equation of state from first principles.

Canonical Ensemble: Now the potential (3) is studied using the canonical ensemble. For this ensemble the system is immerse in a thermal bath with an equilibrium temperature, $T \equiv (k_B \beta)^{-1}$. The other characteristics of the system, mentioned in the microcanonical ensemble, remain the same. For the canonical ensemble we want to determine
the Helmholtz free energy, $F$, through $F = -k_B T \ln Z$, with $Z(T,V,N)$ the partition function; integrating over the momenta, and rescaling the coordinates to the set $\vec{r}_i = (x_i, y_i, z_i)$ in the same manner that in microcanonical ensemble, the Helmholtz free energy can be written as:

$$F = -Nk_B T \ln \left[ \frac{e^V}{N} \left( \frac{k_B TM}{2\pi \hbar^2} \right)^{3/2} \right] - k_B T \phi_N(\eta). \quad (7)$$

The first term is the kinetic contribution to $F$, and the second contains the coordinate part; a new dimensionless variable, defined as $\kappa = GM^3 cN/k_B T J$, is introduced. The integration over the coordinates is expressed using the function $\phi_N(\eta)$, defined using the potential energy per particle $u(\cdot)$, 

$$\psi_N(\kappa) = \ln \int_0^1 \cdots \int_0^1 d^3r_i \exp [\kappa u(\vec{r}_1, \ldots, \vec{r}_N)]. \quad (8)$$

From this expression is possible to find the relevant thermodynamic quantities, like entropy $S$, chemical potential $\mu$, specific heat, etc. In particular, the pressure equation of state $P = -\left( \frac{\partial F}{\partial V} \right)_T, N$, is: $PV/Nk_B T = 1 + L/3N \partial_L \psi_N(\kappa)$. Clearly, the first term in the right-side is the ideal gas term, and the second is the corrective term due the interaction between particles. A function of $\kappa$ and $L$ can be defined in order to simplify equation of state and all the other thermodynamic expressions, therefore $f(\kappa, L)$ is defined as $f(\kappa, L) = 1 + (L/3N) \partial_L \psi_N(\kappa)$. For $\kappa \to 0$, $f(\kappa = 1, L) = 1$ i.e., ideal gas value, independently of $L$. The equation of state is written as

$$\frac{PV}{Nk_B T} = f(\kappa, L). \quad (9)$$

Before we perform the dilute regime for the canonical ensemble, it is important to notice that both entropy (4) and free Helmholtz energy (7), are extensive quantities since they are proportional to $N$ through $\lambda$ and $\kappa$. Also, these new variables are intensive, since they are proportional to $N/L$; related to the dilute thermodynamic limit. In the same manner that the microcanonical ensemble, the dilute regime for the canonical ensemble is obtained; the purpose is to find an analytical expression for the equation of state in the canonical ensemble. Considering again the limit where $N \to 0$, or equivalently $\kappa \ll 1$; under this condition the function $\psi_N(\kappa)$ is written like a power series expansion of $\kappa$ expanding the integrand in (8). With the same assumptions about the interaction between pairs of particles to express $\psi_N(\kappa)$ to second order in $\kappa$; in the limit $N \to \infty$ keeping $\kappa$ constant. Then, the equation of state in this regime, is given by:

$$\frac{PV}{Nk_B T} = 1 + \kappa L \frac{\partial}{\partial L} b_0 + \kappa^2 L \frac{\partial}{\partial L} \left[ \frac{1}{6} b_1 - 6b_0^2 \right] + O(\kappa^3). \quad (10)$$

With this canonical equation of state in the dilute regime, that we obtained from first principles, concludes the study of the self-gravitating system in the canonical ensemble.
DISCUSSION

Long-range systems are characterized by the possible non-equivalence in the thermodynamic information provided by different statistical ensembles. With the previous calculation in the low densities limit (for $N \to \infty$ and keeping fixed $\lambda$ and $\kappa$) of the equations of state for both microcanonical and canonical ensembles (6) and (10), respectively, we are in position to make an analysis of those equations to find if they contain the same information in this approximation. To show the equivalence it is necessary to relate $\lambda$ and $\kappa$ through their definitions; using those relations is easy to find: $\kappa = 3/2\lambda - 9b_0/2\lambda^2 - 9(b_1 - 42b_0)/4\lambda^3 + O(\lambda^{-4})$; however if we want to find equivalence from the microcanonical to the canonical equation of state, instead of a relation like $\kappa = h(1/\lambda)$, a relation like $1/\lambda = q(\kappa)$ is required; with the use of Lagrange inversion theorem it can be achieved. The application of this theorem leads to: $1/\lambda = 2\kappa/3[1 + 2b_0\kappa - 2(10b_0 - b_1/3)\kappa^2 + O(\kappa^3)]$; substituting and simplifying the result can be expressed as

$$\frac{PV}{Nk_B T} = 1 + \kappa L \frac{\partial}{\partial L} b_0 + \kappa^2 L \frac{\partial}{\partial L} \left[ \frac{1}{6} b_1 - 6b_0^2 \right] + O(\kappa^3).$$

(11)

For the canonical equation of state (10); a similar process can be performed in a reverse way, from canonical to microcanonical equation of state. It means that the information provided by both ensembles is equivalent in the dilute regime; however, this result does not ensure the ensemble equivalence out the low densities limit. Finding an equation of state from first principles is remarkable, since most of the proposed equations of state are empirical. Studying both equations we find that both ensembles produce equivalent results in the dilute regime; but it is necessary to ensure that the equivalence is present also out the dilute regime.

REFERENCES