Myers-Perry Solution in 4D

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Abstract.
In this work it is presented a four-dimensional solution with coupled electromagnetic and scalar fields obtained by a dimensional reduction according to the Kaluza-Klein theory from the five-dimensional Myers-Perry solution, that is a 5D spinning black hole. We found that the two five dimensional angular momenta in the four-dimensional solution one of them corresponds to the electromagnetic field while the other is related to the stationarity of the 4D solution.

Keywords: Kaluza-Kein Theory, Myers-Perry Solution, Higher dimensional Black Holes
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INTRODUCTION

Black holes in four-dimensional General Relativity have been the subject of intense research for several decades. In recent years, attention has turned to the study of black holes in higher-dimensions. A five-dimensional asymptotically flat black hole solution was discovered by Emparan and Reall in [1], in addition to the Myers-Perry rotating black hole solution [2]. The Myers-Perry solution has been studied thoroughly [3] as well as the motion of light and particles, quantum radiation, all of them in five-dimensional space-times [4, 5]. The objective of this work is to study solutions in four-dimensions arising from solutions in higher-dimensions, performing a dimensional reduction according to Kaluza-Klein [6]. We address the Myers-Perry 5D black hole and found out that the parameters that in five dimensions represent angular momentum, when down to 4D they are related to electromagnetic field and the 4D angular momentum of the solution.

KALUZA-KLEIN THEORY

The Kaluza-Klein theory is an effective method and the most elegant to unify the electromagnetism with gravity, adding an extra dimension to Einstein equations. The five-dimensional version of the Einstein action is,

$$ S = -\frac{1}{16\pi\hat{g}} \int \hat{R} \sqrt{-\hat{g}} d^4xdy $$

where $y = x^4$ represents the new (fifth) coordinate and $\hat{G}$ is a “five-dimensional gravitation constant”, this way the Einstein equations in five dimensions with vanishing energy-
momentum tensor are

\[ G_{AB} = \tilde{R}_{AB} - \frac{1}{2} \tilde{R} g_{AB} \]  \hspace{1cm} (2)

a convenient parametrization for the metric tensor is

\[ \tilde{g}_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa \phi^2 A_\alpha A_\beta & \kappa \phi^2 A_\alpha \\ \kappa \phi^2 A_\beta & \phi^2 \end{pmatrix} \]  \hspace{1cm} (3)

where \( g_{\alpha\beta} \) is the four-dimensional metric tensor, \( A_\alpha \) the electromagnetic four-potential and \( \phi \) is the scalar field. Klein assumed that the fifth coordinate should be a length like the first three, and assigned it two properties (1) a circular topology \( (S^1) \); and (2) a small scale. Under property (1), any quantity \( f(x,y) \) (where \( x = (x^0,x^1,x^2,x^3) \) and \( y = x^4 \)) becomes periodic; \( f(x,y) = f(x,y + 2\pi r) \) where \( r \) is the scale parameter or “radius” of the fifth dimension. Therefore all the fields can be Fourier-expanded

\[ g_{\alpha\beta}(x,y) = \sum_{n=-\infty}^{\infty} g^{(n)}_{\alpha\beta}(x)e^{i ny/r}, \quad A_\alpha(x,y) = \sum_{n=-\infty}^{\infty} A^{(n)}_\alpha(x)e^{i ny/r} \]  \hspace{1cm} (4)

\[ \phi(x,y) = \sum_{n=-\infty}^{\infty} \phi^{(n)}e^{i ny/r} \]

where the superscript \( (n) \) refers to the \( n \)th Fourier mode. Thanks to quantum theory, these modes carry a momentum in the \( y \)-direction of the order \( |n|/r \). This is where property (2) comes in: if \( r \) is small enough, then the \( y \)-momenta of even the \( n = 1 \) modes will be so large as to put them beyond the reach of experiment. Hence only the \( n = 0 \) modes, which are independent of \( y \), will be observable, as required in Kaluza’s theory.

**MYERS-PERRY SOLUTION**

The five-dimensional Myers-Perry solution corresponds to a five-dimensional spinning black hole. This is an asymptotically flat stationary solution of the vacuum Einstein equations with an event horizon that has the topology of a 3-sphere \( S^3 \). The metric of the five-dimensional Myers-Perry black hole in Boyer-Lindquist coordinates is

\[ ds^2 = \frac{\rho^2}{4\Delta} dx^2 + \rho^2 d\theta^2 - dt^2 + (x+a^2) \sin^2 \theta d\phi^2 + (x+b^2) \cos^2 \theta d\psi^2 \]  \hspace{1cm} (5)

\[ + \frac{r_0^2}{\rho^2} [dt + a \sin^2 \theta d\phi + b \cos^2 \theta d\psi]^2 \]

here

\[ \rho^2 = x + a^2 \cos^2 \theta + b^2 \sin^2 \theta \]  \hspace{1cm} (6)

\[ \Delta = (x+a^2)(x+b^2) - r_0^2 x \]  \hspace{1cm} (7)

where \( r_0, a \) and \( b \) are, respectively, the parameters related to mass and two angular momenta. The angles \( \phi \) and \( \psi \) take values in the interval \([0,2\pi]\) and \( \theta \) take values
in \([0, \pi/2]\); the Killing vectors are \(\partial_t, \partial_\phi\) and \(\partial_\psi\). The black hole horizon is located at \(x = x_+\), where

\[
x_\pm = \frac{1}{2} \left[ r_0^2 - a^2 - b^2 \pm \sqrt{(r_0^2 - a^2 - b^2)^2 - 4a^2b^2} \right]
\]

(8)

the angular velocities \(\Omega_a\) y \(\Omega_b\) are

\[
\Omega_a = \frac{a}{x_+ + a^2}, \quad \Omega_b = \frac{b}{x_+ + b^2},
\]

(9)

and the five-dimensional components of the metric tensor are

\[
\hat{g}_{tt} = \left( \frac{r_0^2}{\rho^2} - 1 \right), \quad \hat{g}_{xx} = \frac{\rho^2}{4\Delta}, \quad \hat{g}_{\theta\theta} = \rho^2,
\]

\[
\hat{g}_{\phi\phi} = \left( \frac{x + a^2}{\sin^2 \theta} + \frac{a^2r_0^2}{\rho^2} \right) \sin^4 \theta, \quad \hat{g}_{t\phi} = \frac{a \sin^2 \theta r_0^2}{\rho^2},
\]

\[
\hat{g}_{\psi\psi} = \left( \frac{x + b^2}{\cos^2 \theta} + \frac{b^2r_0^2}{\rho^2} \right) \cos^4 \theta, \quad \hat{g}_{t\psi} = \frac{b \cos^2 \theta r_0^2}{\rho^2},
\]

\[
\hat{g}_{\phi\psi} = \frac{ab \sin^2 \theta \cos^2 \theta r_0^2}{\rho^2}.
\]

FOUR-DIMENSIONAL MYERS-PERRY SOLUTION

The metric tensor components (10) associated to the Myers-Perry solution have been calculated afterwards by comparison with the metric tensor of Kaluza-Klein theory (3), we identify the expressions of the scalar field and the nonzero components of electromagnetic four-potential

\[
\varphi^2 = \left( \frac{x + b^2}{\cos^2 \theta} + \frac{b^2r_0^2}{\rho^2} \right) \cos^4 \theta,
\]

(11)

\[
A_t = \frac{br_0^2}{\kappa \left[ \rho^2(x + b^2) + b^2r_0^2\cos^2 \theta \right]},
\]

(12)

\[
A_\phi = \frac{abr_0^2 \sin^2 \theta}{\kappa \left[ \rho^2(x + b^2) + b^2r_0^2\cos^2 \theta \right]},
\]

(13)

and the four-dimensional metric tensor in \((x, t, \theta, \phi)\) coordinates is

\[
g_{tt} = \left( \frac{r_0^2}{\rho^2} - 1 \right) + k\varphi^2A_t^2, \quad g_{xx} = \frac{\rho^2}{4\Delta},
\]

\[
g_{\phi\phi} = \left( \frac{x + a^2}{\sin^2 \theta} + \frac{a^2r_0^2}{\rho^2} \right) \sin^4 \theta + k\varphi^2A_\phi^2,
\]

(14)
Once obtained the components of the electromagnetic four-potential, we derive the electromagnetic field tensor

\[ F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \]  \hspace{1cm} (15)

Substituting the expression for \( \rho^2 \) (6) in the electromagnetic field tensor and calculating the partial derivatives we obtained the electromagnetic field tensor components

\[ F_{\rho\phi} = -\frac{br_0^2(\rho^2 + (x + b^2))}{\kappa[\rho^2(x + b^2) + b^2r_0^2\cos^2\theta]^2} \]  \hspace{1cm} (16)

\[ F_{\phi\rho} = \frac{abr_0^2\sin^2\theta(\rho^2 + (x + b^2))}{\kappa[\rho^2(x + b^2) + b^2r_0^2\cos^2\theta]^2} \]  \hspace{1cm} (17)

\[ F_{\phi\theta} = \frac{2br_0^2[(x + b^2)(a^2 - b^2) + b^2r_0^2] \cos\theta \sin\theta}{\kappa[\rho^2(x + b^2) + b^2r_0^2\cos^2\theta]^2} \]  \hspace{1cm} (18)

\[ F_{\theta\phi} = \frac{2abr_0^2[(x + b^2)(\rho^2 + (a^2 - b^2)\sin^2\theta) + b^2r_0^2] \cos\theta \sin\theta}{\kappa[\rho^2(x + b^2) + b^2r_0^2\cos^2\theta]^2} \]  \hspace{1cm} (19)

Remind that the Myers-Perry solution has two angular momenta \( a \) and \( b \), now we can interpret them in 4D. In the particular case with one angular momentum like in [3] we find a stationary and axisymmetrical solution, while the other parameter generates the electromagnetic field. In both cases as a result of dimensional reduction the scalar field (11) is present in the 4D solution.

**Case \( a = 0, b \neq 0, \) with electromagnetic field**

A particular case is to ignore one of the 5D rotations, \( a = 0 \) and \( b \neq 0 \), note than the electromagnetic four-potential components are \( A_\phi = 0, A_\theta = 0 \) and only \( A_\rho \neq 0 \) given by Eq. (12); in this case the four-dimensional metric tensor is diagonal and its components are:

\[ g_{\rho\phi} = \frac{\rho^2}{4\Delta}, \quad g_{\rho\rho} = \rho^2, \quad g_{\phi\phi} = x\sin^2\theta. \]  \hspace{1cm} (20)

And the electromagnetic field tensor components are

\[ F_{\rho\phi} = -\frac{br_0^2(\rho^2 + x + b^2)}{\kappa[\rho^2(x + b^2) + b^2r_0^2\cos^2\theta]^2} \]  \hspace{1cm} (21)

\[ F_{\theta\phi} = -\frac{2b^3r_0^2(x + b^2 - r_0^2) \cos\theta \sin\theta}{\kappa[\rho^2(x + b^2) + b^2r_0^2\cos^2\theta]^2} \]  \hspace{1cm} (22)
Stationary solution when \( a \neq 0, b = 0 \)

Another particular case of this solution is to take \( b = 0 \) and \( a \neq 0 \), the scalar field expression is

\[
\phi^2 = x \cos^2 \theta.
\]  

(23)

note than all components of the electromagnetic four-potential vanish, therefore the electromagnetic field tensor also is zero. The components of the four-dimensional metric tensor are

\[
g_{tt} = \frac{\rho^2}{r_0^2} - 1, \quad g_{xx} = \frac{\rho^2}{4\Delta},
\]

\[
g_{\phi\phi} = \frac{\rho^2(x + a^2) + ar_0^2 \sin^2 \theta}{\rho^2 \csc^2 \theta},
\]

\[
g_{t\phi} = g_{\phi t} = \frac{2ar_0^2 \sin^2 \theta}{\rho^2}, \quad g_{\theta\theta} = \rho^2
\]

(24)

the nondiagonal component \( g_{t\phi} \) being related to an angular momentum.

CONCLUSIONS

The solution with two rotations \((a \neq 0, b \neq 0)\) has non-diagonal components \( g_{\phi t} \), ie, the four-dimensional metric is stationary, however, when we take the parameter \( a = 0 \) these component disappear, so this parameter in four dimension is related to a symmetry of rotation. In contrast, the parameter \( b \) generates a static solution with electromagnetic field \((A_t \neq 0)\), so we can say than in four dimensions, \( b \) is associated to the electromagnetic part of the solution. In both cases we have the scalar field (11) associated with the dimensional reduction.

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