Rigid cosmology: A classical analysis

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Abstract. We explore some cosmological implications obtained in the classical analysis of an effective geometrical action known as rigid cosmology model. We study the associated equation of motion. Within the framework of a minisuperspace, we obtain a generalized Friedmann equation describing the cosmological evolution.

INTRODUCTION

Nowadays, new evidence coming from astrophysics and cosmology has revealed an unexpected picture of the universe. The data coming from different sources, such as the Cosmic Microwave Background Radiation (CMBR) and supernova, seem to indicate that a great percentage of the matter contained in the universe is exotic, i.e., we do not know what that matter is. Puzzling observations do not stop here, a lot of attention has been paid to explain the current cosmic acceleration. General Relativity (GR) can not explain this fact unless dark matter or another sort of exotic matter configurations is appealed.

In the search of an answer, we can consider several alternatives to explain the actual picture of the universe. The concept of a relativistic extended object as a surface immersed in a bulk has increased the interest in physics. In fact, the idea that our universe could be a 3+1 dimensional surface embedded in a higher dimensional spacetime was introduced by Regge and Teitelboim (RT)[1].

In this work, we study the introduction of second order derivatives terms into the original RT model. This type of correction was used before in several contexts, in the framework of the extended objects was used to describe the dynamics taking into account the width of the object[2].

THE RIGID MODEL

We consider here a 3- brane \( \Sigma \), floating in a flat 5-dimensional Minkowski background spacetime with metric \( \eta_{\mu\nu} \). Its trajectory it will be denoted by \( m \) and we can specify such trajectory in the bulk setting \( x^\mu = X^\mu(\xi^a) \), where \( x^\mu \) are the local coordinates for the background spacetime, \( \xi^a \) are local coordinates coordinates for \( m \), and \( X^\mu \) the embedding functions \( (\mu, \nu = 0, 1, ..., 4; a = 0, 1, 2, 3) \). We have derivatives of the parametrization in the induced metric tensor \( g_{ab} = \eta_{\mu\nu} e_\mu^a e_\nu^b = e_a \cdot e_b \) and the extrinsic
curvature of \( m \), \( K_{ab} = -n \cdot D_a e_b \), where \( D_a = e^\mu_a D_\mu \) and \( D_\mu \) is the bulk covariant derivative and \( e^\mu_a = \partial_a X^\mu \) stand for the tangent vectors to \( m \). The action associated to \( m \) in this case is given by

\[
S[X] = \frac{\alpha}{2} \int_m d^{p+1}\xi \sqrt{-g} \mathcal{R} - \int_m d^{p+1}\xi \sqrt{-g} \Lambda + \beta \int_m d^{p+1}\xi \sqrt{-g} K,
\]

where \( \alpha, \beta \) are constants, and \( g \) denotes the determinant of the induced metric \( g_{ab} \). We also included to the action a cosmological constant term \( \Lambda \) and the mean of the extrinsic curvature given by the trace \( \tilde{K} = g^{ab} K_{ab} \) where \( g^{ab} \) denotes the inverse of \( g_{ab} \). The Ricci scalar can be obtained directly from the induced metric \( g_{ab} \), or, in terms of the extrinsic curvature tensor via the contracted Gauss-Codazzi condition \( \mathcal{R} = K^2 - K_{ab}K^{ab} \). The action (1) is invariant under reparametrizations of the worldvolume. The corresponding equations of motion associated to the action (1) are

\[
(\alpha g^{ab} + \Lambda g^{ab})K_{ab} - \beta \mathcal{R} = 0,
\]

where \( g_{ab} = \mathcal{R}_{ab} - \frac{1}{2} \mathcal{R} g_{ab} \) is the worldvolume Einstein tensor, with \( \mathcal{R}_{ab} \) being the Ricci tensor. Note that these equations of motion are second-order in the derivatives of the embedding functions.

**MINISUPERSPACE MODEL**

Now we focus on the case of a 3-brane \( \Sigma \) evolving in a 5-dimensional Minkowski spacetime \( ds^2 = -dt^2 + da^2 + a^2 d\Omega_3^2 \), where \( d\Omega_3^2 \) is the metric of a unit 3-sphere, i.e., \( d\Omega_3^2 = d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \). If

\[
x^\mu = X^\mu(\xi^a) = (t(\tau), a(\tau), \chi, \theta, \phi),
\]

is a parametric representation of the trajectory of \( \Sigma \), we assure that the geometry of the worldvolume generated is that of the FRW case. Here the function \( a(\tau) \) is the well known scale factor. The basis adapted to the worldvolume is given by four tangent vectors \( e^\mu_a \) together with the unit spacelike normal vector

\[
n^\mu = \frac{1}{N}(\dot{a}, i, 0, 0, 0),
\]

where the dot stands for deriviation with respect to \( \tau \). For simplicity in the notation we have introduced the quantity \( N = \sqrt{\dot{a}^2 - \dot{\tau}^2} \). The metric induced on the worldvolume is given by

\[
ds^2 = g_{ab} d\xi^a d\xi^b = -N^2 d\tau^2 + a^2 d\Omega_3^2.
\]

The Ricci scalar associated to (5) and the mean extrinsic curvature are

\[
\mathcal{R} = \frac{6i}{a^2 N^4}(a\dot{a} - \dot{a}^2 + N^2 i),
\]

\[
K = \frac{1}{N^3} (i\ddot{a} - \dot{a}^2) + \frac{3i}{aN}.
\]
The linear dependence in the accelerations \( t(\tau) \) and \( a(\tau) \) of the Ricci scalar and the mean extrinsic curvature is particularly remarkable. In this case the Lagrangian density \( \mathcal{L} = \sqrt{-g}\left(\frac{R}{2} - \Lambda + \beta K\right) \) becomes

\[
\mathcal{L} = \frac{\Psi a_i}{N^3}(a_{\bar{i}} - a_{\bar{i}} + N^2 i) - \frac{a^3 N\Psi}{3} \Lambda - \frac{\beta a^3}{3N^2} \Psi(a_{\bar{i}} - i_{\bar{i}}) + \beta a^2 \Psi i,
\]

where \( \Psi = 3 \sin \theta \sin \chi \). Thus we can reduce our action in the case of spherical configurations to

\[
S = 6\pi^2 \int d\tau L(a, \dot{a}, \ddot{a}, i, \dot{i}),
\]

where the Lagrangian function is given by

\[
L = \frac{ai}{N^3}(a_{\bar{i}} - a_{\bar{i}} + N^2 i) - Na^3 H^2 + \left(\frac{a}{N}\right)^2 I^2 (a_{\bar{i}} - a_{\bar{i}} + 3N^2 i),
\]

where we have introduced the constant quantities \( H^2 := \frac{\Lambda}{3a} \) and \( I^2 := \frac{\beta}{3a} \).

**Ostrogradski-Hamiltonian approach**

Following the Ostrogradski procedure for higher-order derivative systems [3], the highest conjugate momenta to the velocities are

\[
P_t = \frac{\partial L}{\partial \dot{t}} = -\frac{a^2 \dot{a}}{N^3} (i + aN^2),
\]

\[
P_a = \frac{\partial L}{\partial \dot{a}} = a \dot{a} N^3 (i + aN^2).
\]

And the conjugate momenta to the position variables can be written respectively as follows

\[
p_t = \frac{\partial L}{\partial \dot{t}} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{t}} \right) = \frac{ai}{N^3} [\dot{a}^2 + N^2 (1 - a^2 H^2) + 3Na^2 i] =: -\Omega,
\]

\[
p_a = \frac{\partial L}{\partial \dot{a}} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{a}} \right) = \frac{a \dot{a}}{N^3} [-i^2 + a^2 N^2 H^2 - 3Na^2 i].
\]

The appropriate phase space of the system, \( \Gamma := \{ t, a, \dot{a}, \ddot{a}; p_t, p_a, P_t, P_a \} \), has been identified explicitly.
FRIDMANN TYPE EQUATION FOR A RIGID BRANE UNIVERSE

It is convenient obtain it by considering the conserved physical quantities of our model, we have that (13) can be written as

\[ x^3 + \frac{h}{H} x^2 - x + \frac{\Omega}{a^4 H^3} = 0, \]  

(15)

where we have defined \( x = \frac{\dot{t}}{NaH} \), now if we consider the change of variable \( x := y - h \), we get

\[ y_i^3 - (1 + 3\rho^2)y_i + \rho(1 + 2\rho^2) + \frac{2}{a^4} \sigma = 0, \]

(16)

where \( \rho := \frac{h}{3H} \) and \( \sigma := \frac{\Omega}{2H^3} \). This type of equation is known as incomplete cubic equation. We find that the physical solution to equation (16) is

\[ y = 2\sqrt{\rho^2 + \frac{1}{3} \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{1}{2} \frac{\rho(1 + 2\rho^2) + \frac{2\sigma}{a^4}}{\sqrt{\left( \frac{1}{3} + \rho^2 \right)^3}} \right) \right)}. \]  

(17)

Adopting the usual cosmic gauge, \( N = 1 \), and inserting the expression (17) into the definition for \( x \) and squaring we get the Friedmann type equation

\[ \dot{a}^2 + \varphi(a) = -1, \]  

(18)

where we have an effective potential

\[ \varphi(a) = -a^2 H^2 \left[ 2\sqrt{\rho^2 + \frac{1}{3} \cos \left( \frac{1}{3} \cos^{-1} \left( \frac{1}{2} \frac{\rho(1 + 2\rho^2) + \frac{2\sigma}{a^4}}{\sqrt{\left( \frac{1}{3} + \rho^2 \right)^3}} \right) \right)} - \rho \right]^2, \]  

(19)

we show the behaviour of the potential in the figure (1). Note that with positive values for the rigidity term \( \beta \) we have a faster expansion of our universe.

CONCLUSIONS

We have studied a simple geometrical model which gives and accelerated universe, in dependance of the nature of the rigidity \( \beta \), because it acts as producer or preventer of this acceleration. We still have several possibilities in the description of the cosmic evolution of this model. We used the Ostrogradski formalism to identify the conserved energy of the model, which is important in our analysis.

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FIGURE 1. Effective potential describing possible trajectories for $\Sigma$.

REFERENCES