Relativistic surface with maximal acceleration

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Abstract. We propose an effective model, geometric, to describe the dynamics of extended objects with maximal acceleration, which evolve in a Minkowski spacetime. The effective model involves the extrinsic curvature of the trajectory generated by the object during its evolution. The Lagrangian describing this theory is of second order in derivatives and thus the equations of motion are of fourth order in the coordinates. We show that in the case of codimension one, the equations of motion resembles a Klein-Gordon type equation. For illustration, we study the dynamics of a (3+1) spherical surface, having an accelerated expansion where there is a noticeable maximal acceleration.

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INTRODUCTION

Many astrophysical observations have clearly confirmed that our universe is expanding rapidly [1]. This cosmic acceleration can not be explained by any of the fundamental forces of the standard model, therefore, is to become one of the greatest challenges of theoretical physics today. An alternative is intensively studied in the context of general relativity (GR), which proposes an exotic component characterized by a negative pressure called dark energy, which is used to explain the observed phenomena [2]. The simplest candidate for dark energy, consistent with current observations, is the cosmological constant. Thus, many dynamic scalar fields such as quintessence, K-essence, phantom, etc, to name a few, have been proposed as an alternative to explain the dark energy. But until now, the nature of the dark energy remains unclear.

Moreover, the idea of Brane world universes has attracted interest in physics because of its wide range of applications, especially in cosmology, as they offer a novel approach to our understanding of the universe and its evolution. The idea that our observable universe (3+1) dimensional immersed in a higher dimensional space has many years [3, 4].

On the other hand, the geometric models that describe relativistic membranes have been of great interest in different contexts of science because of its potential to describe physical systems modeled by means of extended objects. For example, Dirac tried to model the electron by a charged spherical surface [5], a geometric alternative to model the confinement of quarks in hadrons was suggested considering a relativistic string with monopoles at the ends [6], in the Euclidean context many effective models to describe lipid membrane are based on the concept of surface and curvature [7]. In the cosmological context, in order to model the evolution of our universe by means
of relativistic surfaces immersed in a 4-dimensional spacetime into a higher dimension.

We propose an effective model to describe relativistic surfaces present a bound on the acceleration of its evolution. This model features a non-polynomial dependence on the extrinsic curvature of the surface, which in turn becomes a theory of second order derivatives of the field variables. In the case of relativistic assume that our surface is of codimension one, the corresponding equation of motion will be similar to an equation of Klein-Gordon type. Our approach is illustrated considering a spherical surface. Solving the equation of motion for the acceleration of the surface there is a bounded acceleration. The paper is organized as follows: Section II, we present the theoretical model and calculate maximal acceleration equations of motion. In section III we analyze the case of a spherical surface at maximal acceleration. We conclude in Section IV with comments and prospects.

**MODEL WITH MAXIMAL ACCELERATION**

Consider an extended object \( \Sigma \), of dimension \( p \), evolving into an \( N \)-dimensional Minkowski spacetime, with metric \( \eta_{\mu\nu} \), \((\mu, \nu = 0, 1, \ldots, N - 1) \). Its trajectory or world volume \( m \), of dimension \( p + 1 \) is described by the parametrization \( x^\mu = X^\mu(\xi^a) \), where \( x^\mu \) are the local coordinates for the background spacetime, \( \xi^a \) are the local coordinates for the world volume and \( X^\mu \) are the embedding function \((a, b = 0, 1, \ldots, p) \). Denote the tangent vectors to \( m \), as \( e^\mu_a = \partial_a X^\mu \). In this context, we introduce \( N - p - 1 \) unit normal vectors to the world volume denoted as \( n^i_\mu \) \((i = 1, 2, \ldots, N - p - 1) \). Normal vectors to \( m \) are implicit defined by \( n^i \cdot e_a = 0 \) with normalization \( n^i \cdot n_j = \delta_{ij} \). Thus the vectors \( \{e^\mu_a, n^i_\mu\} \) form an orthogonal basis on the world volume.

Nesterenko et al [8], proposed a one-dimensional geometric model for a relativistic particle with a bound on acceleration. Generalizing the action to the context of branes, we construct a geometric model with the intention to describe the dynamics of extended objects with bound acceleration

\[
S[X^\mu] = \alpha \int d^{p+1}\xi \sqrt{-g} \sqrt{M_0^2 - \tilde{K}^i K_i}, \tag{1}
\]

where \( \alpha \) is a constant with dimension \([L]^{1-p} \), \( M_0 \) is a constant with dimension \([L]^{-2} \); \( g \) denotes the determinant of the induced metric \( g_{ab} = e_a \cdot e_b \). The term \( \tilde{K}^i = g^{ab} K_{ab}^i \) is the half extrinsic curvature, where \( K_{ab} = -n \cdot \nabla_a e_b \) is the extrinsic curvature of the path \( m \). The main symmetry behind the model (1) is the invariance under world volume reparameterizations.

A geometric object quite useful in the description of an extended object is the stress tensor preserved \( f^{\mu a} \), which provides information on the dynamics of the object [9]. For the system under consideration the stress tensor is

\[
f^{\mu a} = \alpha \left( g^{ab} \sqrt{M_0^2 - \tilde{K}^i K_i + \tilde{K}^i K^{abi}} \right) e^\mu_b - \alpha g^{ac} \left( \nabla_c \tilde{K}^i \right) n^{\mu i}, \tag{2}
\]

where \( \tilde{K} := \frac{K}{\sqrt{M_0^2 - K^2}} \), to obtain the equations of motion, we only look at the normal projection of the conservation law \( \nabla_a f^{\mu a} = 0 \), where \( \nabla_a \) is the covariant derivate compatible...
with the metric $g_{ab}$. The resulting equation of motion are

$$\ddot{\tilde{K}}^i + \dot{\vartheta}^j \tilde{K}_j = 0,$$  \hspace{1cm} (3)

where

$$\dot{\vartheta}^i = \left[ K^{ab} K^i_{\ j} + \left( M_0^2 - K^{l} K_l \right) \delta^i_j \right],$$

Besides considering the D’Alembertian operator $\tilde{\Delta} = g^{ab} \tilde{\nabla}_a \tilde{\nabla}_b$. These equations are fourth order in derivatives of the coordinates. We note that in the case of codimension one $(i = 1)$ and to make use of the Gauss-Codazzi equation $\mathcal{R} = K^2 - K_{ab} K^{ab}$, we have the equation of motion (3) are simply

$$\left[ \Delta + (M_0^2 - \mathcal{R}) \right] \tilde{K} = 0,$$  \hspace{1cm} (4)

where $\mathcal{R}$ is the Ricci scalar world volume $m$, we see that (4) is similar to a Klein-Gordon type equation$^1$.

**SURFACE WITH MAXIMAL ACCELERATION**

Now restrict our model to a (3+1) brane, $\Sigma$ spherical, evolving in a fixed Minkowski 5-dimensional background spacetime [10], with metric $ds^2 = -dt^2 + da^2 + a^2 d\Omega_3^2$, where $d\Omega_3^2$, is the metric of the unitari 3-sphere $d\Omega_3^2 = d\xi^2 + \sin^2 \chi d\theta + \sin^2 \chi \sin^2 \theta d\phi$.

According to current cosmology and for simplicity we choose the function $\sin^2 \chi$ in $d\Omega_3^2$ to consider a closed universe. By using

$$\chi^\mu = X^\mu(\xi^a) = (t(\tau), a(\tau), \chi, \theta, \phi),$$  \hspace{1cm} (5)

parametric representation of the path of $\Sigma$, is to be the world volume geometry is similar to that generated by a spacetime of Friedmann-Robertson-Walker $(\mu, \nu = 0, 1, \ldots, 4; a, b = 0, 1, \ldots, 3)$. According to standard cosmology $a(\tau)$ is the scale factor. For the base appropriate to the world volume, the tangent vectors $e^\mu_a$ are

$$e^\mu_t = (i, \dot{a}, 0, 0, 0), \hspace{1cm} e^\mu_a = (0, 0, 1, 0, 0), \hspace{1cm} e^\mu_\chi = (0, 0, 0, 1, 0), \hspace{1cm} e^\mu_\theta = (0, 0, 0, 0, 1),$$

and the unit normal vector is spacelike $n_\mu = \frac{1}{N}(\dot{a}, i, 0, 0, 0)$, where the dot denotes derivative with respect to $\tau$ and where we define the function $N := \sqrt{\dot{a}^2 - a^2}$. The normal vector with the tangent vectors allow us to obtain the induced metric in the world volume

$$ds^2 = g_{ab} d\xi^a d\xi^b = -N^2 d\tau^2 + a^2 d\Omega_3^2,$$  \hspace{1cm} (6)

the spatial components of this metric correspond to the metrics associated with $\Sigma$, where it is described by embedding functions (5) in the world volume.

$^1$ Moreover (4) is similar to equation of motion of scalar field non minimally coupled, $(\Box - \xi \mathcal{R}) \phi = 0$.
The Ricci scalar associated with the metric (6) is
\[ \mathcal{R} = \frac{6i}{a^2N^4}(a\ddot{a} i - a\dot{a}^2 + N^2 i). \] (7)

Similarly, the extrinsic curvature tensor associated to the metric (6), the nonzero components are:
\[ K^{\tau \tau} = \frac{i^2}{N^3} \frac{d}{d\tau} \left( \frac{\dot{a}}{i} \right), \quad K^{\chi \chi} = K^{\theta \theta} = K^{\phi \phi} = \frac{i}{aN}. \] (8)

Substituting the values of the extrinsic curvature (8), the expression for \( \tilde{K} \) and with the Ricci scalar in (4), we have after a direct calculation, the equations of motion for the surface with a bound on the acceleration
\[ \omega = \frac{a^2 (1 + \dot{a}^2)}{\{a^2 M_0^2 (1 + \dot{a}^2) - (3 + 3 \dot{a}^2 + a\ddot{a})^2\}^{3/2}} \]
\[ \{a^4 M_0^4 (1 + \dot{a}^2) - (1 + \dot{a}^2)(3 + 3 \dot{a}^2 + a\ddot{a}) [a^2 M_0^2 (6 + 3 \dot{a}^2 + a\ddot{a}) - 3(3 + 3 \dot{a}^2 + a\ddot{a})^2] \]
\[ -a^4 \dot{a} M_0^2 \left[ \frac{3\dot{a}}{a^2} (a\ddot{a} - 1 - \dot{a}^2) - \frac{\dot{a}\ddot{a}^2}{1 + \dot{a}^2 + \ddot{a}} \right] \}, \] (9)

where \( \omega \) is the internal energy of the (3+1) brane. Solving the equation numerically with the appropriate parameters, we obtain the behavior of the acceleration (see fig. 1), which tells us that the brane \( \Sigma \), is accelerating but has a bound in the acceleration, related to the constant \( M_0 \).

FIGURE 1. Behavior of the acceleration vs. time. The acceleration starts to reach an asymptote.
CONCLUSION

We present a geometric model for moving surfaces and have a bound on acceleration. We exemplify our model with a spherical surface. A simple example but with physical relevance that we can describe with this model is a soap bubble. The bubble begins to grow (radially) rapidly but reaches an asymptote or bound in acceleration, then we analyze the stability of the bubble and the tension of it, the constant $\alpha$ in (1) is the tension of the brane in our model.

It should be noted that the proposed model is still a preliminary model which requires a deeper theoretical study, despite the above fact motivates us to see results as the behavior of Figure 1.

Since we control the model, now our next step is the application of the model in the context of cosmology in extra dimensions as a dynamic alternative to explain the accelerating expansion of the Universe [11].

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REFERENCES

11. A. Cervantes, E. Rojas and Cuauhtemoc Campuzano In progress